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# ABOUT UNITS AND DIMENSIONS

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Abstract

Present-day views on the use of units and dimensions are brought forth in a logical presentation and, some of the far too numerous defects in the current litterature are critisized.

## 1. *Introduction*

The requirement for perfection and consistency of science and technology involves the need of using mathematical expressions, tables, equations and so on in clarifying causal connections with different arguments and phenomena. To be most effective the treatment of quantities should be performed in a unified manner, which is complete, clear and practical at the same time. These different demands cannot be completely fulfilled by one single method. Therefore it is necessary either to use different methods in parallel or to accept the one method which most completely takes the different arguments into account. Calculation with pure numbers was the only method used at the early stages of science and technology. The trend to generality led to the invention of unit systems, *e.g.* the cgs system and the MKSA system. At the same time the fundamental ideas of the quantity calculation were realized and since then this expedient has obtained a more and more widening practice. As a matter of fact, the calculation with numbers which still is used in some extent, may be considered as a relict from older times.

The advent of quantity calculation has made the treatment consistent and brought theory and praxis closer to each other.

This article was prompted by the many inaccuracies and mistakes in the use of units found in litterature. Some examples will be given later on.

## 2. *General*

A unit is any quantity of exact magnitude used in measurements. Most standard units are defined by international conventions. A unit has then two characteristics, it expresses the kind of the quantity to be measured and it has a certain magnitude. Here are some examples of units: s, kg, torr, g/cm$^3$, nautical mile.

A dimension is similar to a unit, but it has no magnitude. It hence characterizes the type of quantity only. The dimensions corresponding to the units mentioned above are: time, mass, pressure, density, length. Mostly the words used for dimensions are replaced by single letters or mathematical expressions. So the dimensions are respectively: $T$, $M$, $L^{-1} MT^{-2}$, $ML^{-3}$, $L$. If the dimension of a quantity, given as a single letter or as a mathematical expression, is desired, the expression itself is inserted in brackets. For example $[a] = [\omega^2 r] = [\omega]^2[r] = LT^{-2}$. Sometimes, the brackets are used in a similar fashion for the determination of units as well.

In mechanics it is necessary to choose three quantities as fundamental. Usually they are length, mass and time. The choice is indeed optional provided that the three quantities are independent of each other. For this reason, there are several unit systems existing. In other branches of science some more fundamental quantities are useful when not always necessary. So the electric current and temperature may be taken as fundamental quantities. In chemistry and related sciences, it is practical to take mole or val as a fundamental unit. For example, the determination of the dimension of the universal gas constant from the equation of state of gases gives

$$[R] = \left[\frac{p\,V}{\mu\,T}\right] = \frac{[W]}{[\mu]\,[T]} = \frac{L^2MT^{-2}}{A\Theta}\ ,$$

where

$R$ = universal gas constant
$p$ = pressure
$V$ = volume
$W$ = work (energy)
$[\mu] = A =$ dimension of mole
$[T] = \Theta =$ dimension of temperature.

### 3. *Factors of magnification*

Many decimal units are composed of a coefficient part and a unit part, *e.g.* km, ml, pF. Such derived units are thus products of magnification factors and basic units. The magnification coefficients are powers of ten with integral positive or negative exponents. The factors of magnification are given in Table 1 (*cf.* SUOMEN STANDARDISOIMISLAUTAKUNTA, [5]).

Table 1. Magnification coefficients used in connection with decimal units.

| Name | Denotation and value |
|---|---|
| tera | T $= 10^{12}$ |
| giga | G $= 10^{9}$ |
| mega | M $= 10^{6}$ |
| kilo | k $= 10^{3}$ |
| hecto | h $= 10^{2}$ |
| deca | D $= 10$ |
| deci | d $= 10^{-1}$ |
| centi | c $= 10^{-2}$ |
| milli | m $= 10^{-3}$ |
| micro | $\mu = 10^{-6}$ |
| nano | n $= 10^{-9}$ |
| pico | p $= 10^{-12}$ |

As a rule, only one coefficient or magnification factor is used in one quantity. Nowadays, there seems to be a tendency to use these factors in connection with most decimal units. For example 1 nm $= 10^{-3}$ $\mu$m $= 10^{-6}$ mm $= 10^{-9}$ m. It is not allowed to write m$\mu$F but nF should be used instead.

### 4. *Use of units*

*Graphs.* In mathematics, it is customary to write the variables near the arrowheads indicating the positive directions of the axes. (Fig. 1).

The numerical values corresponding to points on the axes are written in the immediate vicinity of them. A particular point of the abscissa, say, may be identified by writing

$$x = a \; .$$

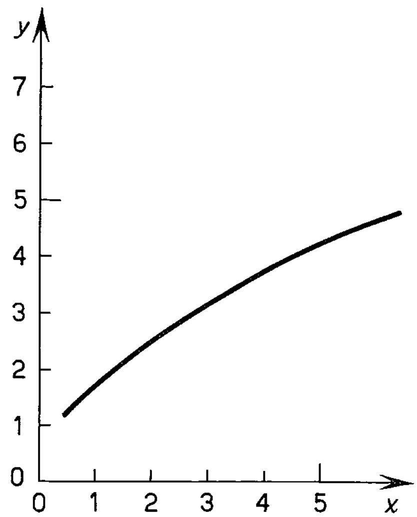


Fig. 1.

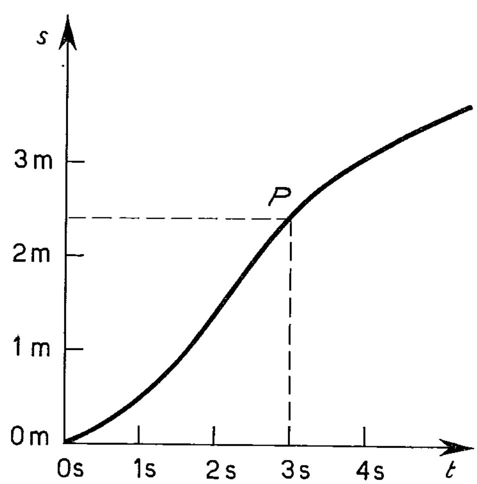


Fig. 2.

The same idea may be transferred into other sciences where not only numerical values, but quantities of different kind are used. In order to write a similar equation for this case, all quantities $a$ and $x$ must be of the same kind. This can be satisfied in different ways. In Fig. 2 the point $P$ on the curve may be identified as

$$t = 3 \text{ s},$$

$$s = 2{,}4 \text{ m}.$$

In this case the quantities themselves (*i.e.* the numerical values times units) are given on the axes. In Fig. 3 the maximum point of the curve is found to be at

$$\frac{\lambda}{10^3 \text{Å}} = 5{,}7$$

or

$$\lambda = 5700 \text{ Å}.$$

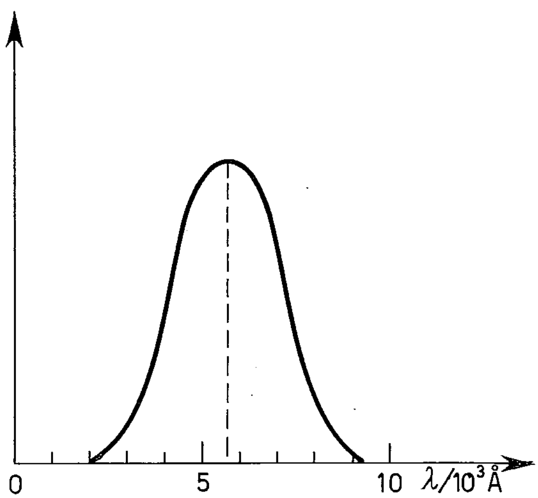


Fig. 3.

In this case, the figures on the axis are pure numbers obtained by dividing corresponding quantities by a suitable constant of the same dimension. This example gives a rather practical method in plotting curves with cumbersome numerical values.

When double scales are used on a certain axis, care should be exercised to write the general notation in the same row or column as the corresponding numbers.

## An example of collapsed logic

To the numbers along the ordinate axes of Fig. 4 (HOLMBOE — FORSYTHE — GUSTIN,[2]) no units have been assigned. They should be pressures expressed in centibars. The notation $\ln p$ should mean that the scale is logarithmic. These defects can be remedied as indicated in Fig. 5. The figures beneath the pressure scale correspond to the general notation $p$/cb in a manner explained above. The temperature notations have been similarly changed. The fact that the pressure scale is logarithmic, is immediately seen from the figure, but may be stressed in the subscript. According to the notation in Fig. 4 the logarithm of the pressure has entered logarithmically.

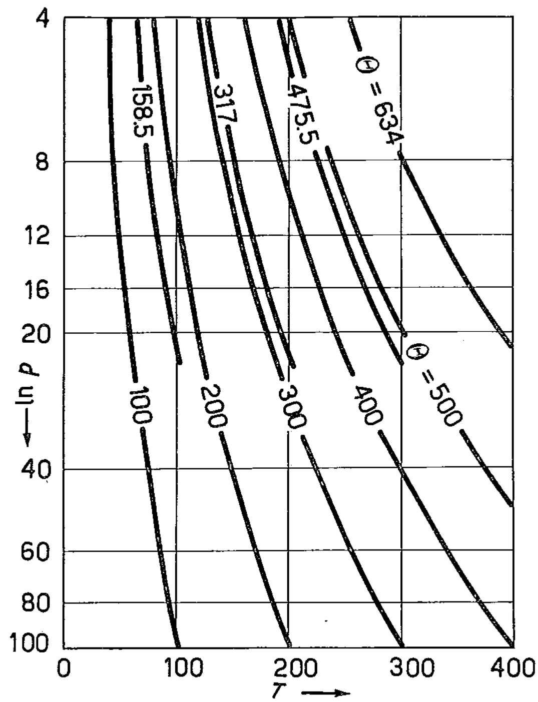


Fig. 4. Original representation of a set of curves, emagram.

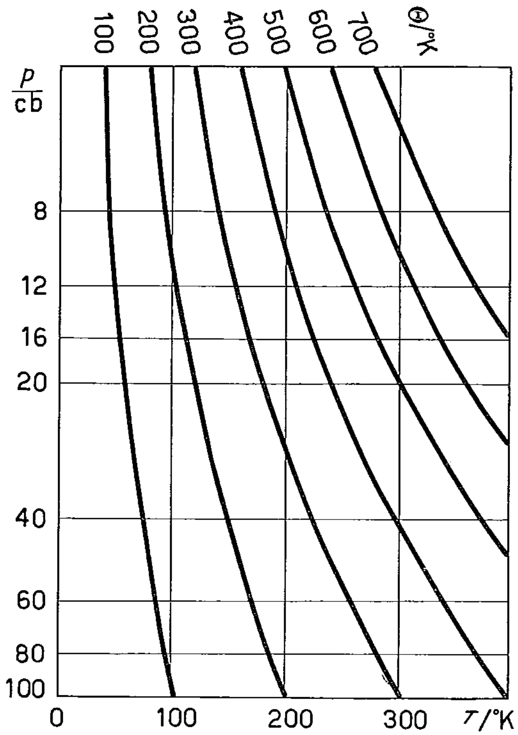


Fig. 5. Corrected representation of the curves of Fig. 4. Note: the pressure scale is logarithmic.

*Tables.* The method explained in the last example for graphs is practical also for tables. In Table 2 the density of destilled water is given as a function of temperature. (*cf.* KOHLRAUSCH, [3]).
For example, from

$$t/°\mathrm{C} = 12$$

$$\varrho/\mathrm{g\ cm^{-3}} = 0{,}999525$$

it follows that

$$t = 12°\mathrm{C}$$

$$\varrho = 0{,}999525\ \mathrm{g\ cm^{-3}}\,.$$

It is customary to divide the table area into appropriate sections to separate the different parameters.

Table 2. Dependence of the density $\varrho$ of destilled water on temperature $t$.

| $t$/°C | $\varrho$/g cm$^{-3}$ |
|---|---|
| 0 | 0,999868 |
| 2 | 0,999968 |
| 4 | 1,000000 |
| 6 | 0,999968 |
| 8 | 0,999876 |
| 10 | 0,999728 |
| 12 | 0,999525 |
| 14 | 0,999271 |
| 16 | 0,999970 |
| 18 | 0,998623 |
| 20 | 0,998232 |

Table 3. Density $\varrho$ of dry air as a function of air temperature $t$ and pressure $b$.

| | $t$/°C | −33 | −13 | +7 | +27 |
|---|---|---|---|---|---|
| $b$/mb | $\varrho$/g m$^{-3}$ | | | | |
| 960 | | 1394 | 1287 | 1195 | 1116 |
| 980 | | 1423 | 1314 | 1220 | 1139 |
| 1000 | | 1452 | 1341 | 1245 | 1162 |
| 1020 | | 1482 | 1368 | 1270 | 1185 |
| 1040 | | 1511 | 1396 | 1295 | 1209 |

For the tabulation of a function of two variables the scheme of Table 3 is practical. (*cf.* Zubov, [7]). The value is to be found at the intersection of corresponding argument rows and columns. This is true for the notations as well. Also here, the numbers and notations belonging to different parameters, have been separated by lines into different sections. For example the triple

$$b/\text{mb} = 1020$$
$$t/°\text{C} = -13$$
$$\varrho/\text{gm}^{-3} = 1368$$

gives

$$b = 1020 \text{ mb}$$
$$t = -13°\text{C}$$
$$\varrho = 1368 \text{ g m}^{-3}.$$

*Equations.* In most equations of science and technology letters are used instead of quantities. Equations of this type are called quantity equations. Because a quantity may be considered as a product of a numerical value and a unit, the calculation with units implies that numerical values and units are equally rated. The quantity equations (*cf.* SUOMEN STANDARDISOIMISLAUTAKUNTA,[4]) are contrasted against numerical equations in which no units are incorporated but only numerical values of the quantities expressed in certain units. The quantity equations are always valid no matter how the units are chosen. Furthermore, they contain more information because of the units included. The numerical equations again, are valid with certain restrictions only, say, the units used belong to a specified unit system. Generality is the primary reason, why the quantity equations are in common use. An example will clarify the evaluation of a quantity from a quantity equation.

The density of dry air is to be found when the air pressure is 1020 mb and the temperature 20°C.

From the equation of state for gases we find

$$\varrho = \frac{pM}{RT} . \tag{1}$$

The parameters are the following:

$\varrho$ = density of the air

$p$ = 1020 mb = air pressure

$M$ = 28,97 g mol$^{-1}$ = apparent molecular weight

$R$ = 8314 J mol$^{-1}$°K$^{-1}$ = universal gas constant

$T$ = 293,2 °K = temperature of the air in the Kelvin scale.

In addition, we use the following relationships between units:

$$1\ \mathrm{b} = (10^6\ \mathrm{dyn\ cm^{-2}} =)\ 10\ \mathrm{N\ cm^{-2}}$$

$$1\ \mathrm{J} = 1\ \mathrm{Nm}\,.$$

From (1) we then have

$$\varrho = \frac{1020\ \mathrm{mb} \cdot 28{,}97\ \mathrm{g\ mol^{-1}}}{8314\ \mathrm{J\ mol^{-1}\ {}^\circ K^{-1}} \cdot 293{,}2\ {}^\circ\mathrm{K}} \times \frac{10\ \mathrm{N\ cm^{-2}}}{1\ \mathrm{b}} \times \frac{\mathrm{J}}{\mathrm{Nm}}$$

$$= \frac{1{,}020 \cdot 28{,}97 \cdot 10}{8314 \cdot 293{,}2} \frac{\mathrm{g\ cm^{-2}}}{\mathrm{m}}$$

$$= 1{,}212 \cdot 10^{-3}\ \mathrm{g\ cm^{-3}}\ .$$

The elimination of unwanted units has been performed by using additional factors each of them being equal to a quotient of two equal quantities, thus multiplying with 1 each time.

If an equation shall be used several times, it is unnecessary to perform the same calculations repeatedly. One may then transform the equation into such a form that unwanted units disappear after the substitution of the known quantities. For example, let us assume that the wind speed is given in ft/s, the time in hours and the distance travelled by the wind is wanted in nautical miles. This situation is governed by the equation

$$s = vt\,. \tag{2}$$

We proceed as follows (*cf.* Suomen Standardisoimislautakunta, [4]):

$$s = \frac{v}{\mathrm{ft/s}} \cdot \mathrm{ft/s} \cdot \frac{t}{\mathrm{h}} \cdot \mathrm{h}\,. \tag{3}$$

Now

$$\mathrm{ft/s} \cdot h = 3600 \ \mathrm{ft} = 3600 \cdot 0{,}3048 \ \mathrm{m}$$

$$= \frac{3600 \cdot 0{,}3048}{1852} \ \mathrm{nm} = 0{,}5925 \ \mathrm{nm}\,.$$

From (3) we get

$$s = 0{,}5925 \ \frac{v}{\mathrm{ft\ s^{-1}}} \ \frac{t}{\mathrm{h}} \ \mathrm{nm}\,. \tag{4}$$

For

$$v = 25 \ \mathrm{ft/s}$$

$$t = 24 \ \mathrm{h}$$

we then have simply

$$s = 0{,}5925 \cdot 25 \cdot 24 \ \mathrm{nm}$$

$$= 355{,}5 \ \mathrm{nm}\,.$$

Still the equation (4) is general in the sense that it is valid for any set of units imaginable, but it is practical only for units indicated.

The numerical equations are used in science and technology to a certain extent. They are relations between numerical values of quantities. Therefore they are valid in specific cases only. In using numerical equations it is necessary to specify the units exactly. For example, if

$s$ km = distance travelled by wind

$v$ m/s = wind speed

$t$ h = corresponding time,

then

$$s = 3{,}6\ vt\,.$$

If we choose instead

$s$ nm = distance travelled by wind

$v$ ft/s = wind speed

$t$ h = corresponding time,

then the equation reads

$$s = 0{,}5925\ vt\,.$$

The lack of generality makes the numerical equations less suitable for theoretical treatment. Furthermore they are susceptible for defects regarding units, misinterpretations and actual errors. For this reason, they hardly can be recommended even for numerical work.

## Examples of defects and mistakes

Evaporation from a water surface can be calculated approximately by the following formula (*cf.* Witting, [6]):

$$Q = k\sqrt{w}\ (1 + \alpha t)\ (e_0 - e)$$

where

$Q$ = evaporation in 24 hours

$w$ = average wind speed

$t$ = temperature

$e_0$ = pressure of water vapor at water surface

$e$ = pressure of water vapor in the air

$k$ = 0,45.

Finding of the correct units in this case is actually a guesswork. Many authors have the feeling that units can be omitted or added almost at will in formulas. For example, the density of water is many times expressed as $\varrho = 1$ instead of $\varrho = 1$ g $cm^{-3}$.

The formula (Gutenberg, [1]) =

$$C = 20{,}06 \sqrt{T} \ (\text{m sec}^{-1})$$

for the calculation of the speed of sound at the pressure of one atmosphere, when the absolute temperature of air is $T$ °K is understandable although not quite exact.

### 5. *Significance of dimensions in checking*

It is clear that both sides of an equation must have the same dimension. Also, terms of a sum have the same dimension in general. (One exception may be mentioned. When in a sum there are terms that add up to zero, the different terms need not necessarily be of the same kind. This is indeed a rare case. Therefore we may leave it out of consideration.) When, for example, a factor has been dropped, the missing factor may be found comparing dimensions of corresponding terms. We give some examples. To check the validily of the capillarity equation

$$h = \frac{2\alpha}{\varrho g r}. \tag{5}$$

The dimensions of the quantities are as follows.

$$[h] = L$$
$$[\alpha] = MT^{-2}$$
$$[\varrho] = ML^{-3}$$
$$[g] = LT^{-2}$$
$$[r] = L.$$

The dimension of the right hand side is thus

$$\left[\frac{2\alpha}{\varrho g r}\right] = \frac{MT^{-2}}{ML^{-3}LT^{-2}L} = L.$$

Because the left hand side has the same dimension, equation (5) has passed this test.

It is not always necessary to resort to the dimensions themselves, rather, it may be advantageous to use auxiliary equations, especially, if the dimensions of some quantities are not known. We may surmise, for example, that the last term in equation

$$\frac{\partial \mathbf{v}}{\partial t} = - \frac{\nabla p}{\varrho} + \mu \Delta \mathbf{v}$$

is wrong.
Here

$\mathbf{v}$ = speed of a fluid
$t$ = time
$p$ = pressure
$\varrho$ = density
$\mu$ = eddy viscosity coefficient.

The ratio of any two terms should be dimensionless. Assume, that we do not remember the dimension of $\mu$, but instead the defining equation

$$F = \mu A \frac{dv}{ds}, \tag{6}$$

where

$F$ = force
$A$ = surface area
$ds$ = separation of two surfaces in which the velocity difference is equal to $dv$.

When we notice in addition that

$$[\nabla p] = \left[\mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z}\right] = \left[\frac{\partial p}{\partial x}\right] = \left[\frac{p}{x}\right] = [p]\,L^{-1},$$

$$[\Delta \mathbf{v}] = \left[\frac{\partial^2 v}{\partial x^2}\right] = \left[\frac{v}{x^2}\right] = [v]\,L^{-2},$$

and

$$F = pA,$$

we find for the dimension of the ratio of the two terms on the right hand side of the equation:

$$\left[\frac{\mu \Delta \mathbf{v}}{\nabla p/\varrho}\right] = \left[\frac{\varrho \mu \Delta \mathbf{v}}{\nabla p}\right] \cdot \left[\frac{F}{\mu A\ dv/ds}\right] \cdot \left[\frac{pA}{F}\right]$$

$$= \frac{[\varrho]\,[v]\,L^{-2}\,[p]}{[p]\,L^{-1}\,[v]L^{-1}} = [\varrho] \neq 1.$$

The discrepancy of dimensions of the two terms checked implies that the last term possibly should be divided by a quantity of dimension[$\varrho$]. In fact, the factor missing is just $1/\varrho$ .

It is clear that the purpose of dimensional checking will be fulfilled also when units are used instead of dimensions. In many cases this is practical because the units of the quantities are mostly at hand.

## 6. *Summary*

Exact mathematical treatment of quantities is of importance in making the theoretical treatment or practical calculations clear-cut and consistent. This is true not only for equations and formulas but for figures and tables as well. The use of dimensions presents an excellent method for checking the validity of equations. Therefore this means should be generally recognised among scientists and technical people.

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