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ON A GENERALIZATION OF THE THOMSON-HASKELL MATRIX METHOD FOR DETERMINING LOVE WAVE DISPERSION

by

P. Saastamoinen

Institute of Seismology, University of Helsinki

Abstract

The Thomson-Haskell matrix method will be extended to the case where the elastic parameters $\mu(z)$ and $\varrho(z)$ can be in the layers and in the halfspace certain simple functions. Then the method is applied to the case where $\mu(z)$ and $\varrho(z)$ are piecewise linear functions or constants.

The problem is to calculate the eigenvalues of the Love wave operator consisting of the differential expression

$$\frac{d}{dz}\left(\mu(z)\frac{dv}{dz}\right) + k^2 \,(c^2\, \varrho(z) - \mu(z)\, v = 0 \tag{1}$$

and the boundary conditions

$$\begin{aligned} \mu(z)\frac{dv}{dz} &= 0\,;\ z = 0 \\ v \to 0\,;&\ z \to \infty \end{aligned} \tag{2}$$

The method developed by W. Thomson [3] and N. Haskell [1] for calculation of the spectrum of the previous operator uses piecewise constant values of $\mu(z)$ and $\varrho(z)$. For some purposes, however, it may prove useful to make a slight generalization of the above method, using for $\mu(z)$ and $\varrho(z)$ piecewise simple functions. This means that we substitute $\mu(z)$ and $\varrho(z)$ in the j:th layer, for every value $j = 1, \ldots n + 1$ where $n + 1$ means the halfspace, by approximations for which it is possible to solve the equation (1) in closed form.

There are $(n + 1)$ differential equations like (1), one in each layer and one in the halfspace. In order to make the problem determinate, the boundary values (2) and the following conditions (the continuity of v and $\mu(z)\frac{dv}{dz}$ on the j:th boundary) are used:

$$v_{(j-1)} = v_{(j)}\,; \text{ on the } (j-1)\text{:th boundary (see Fig. 1) and}$$
$$\mu_{(j-1)}\frac{dv_{(j-1)}}{dz} = \mu_{(j)}\frac{dv_{(j)}}{dz}\,; \text{ on the } (j-1)\text{:th boundary (see Fig. 1)} \tag{3}$$

In (3) $v_{(j-1)}$ and $v_{(j)}$ are the solutions of (1) for the $(j-1)$:th and (j):th layers respectively. $\mu_{(j-1)}$ and $\mu_{(j)}$ are the approximated shear modules in the above-mentioned layers. When two linearly independent solutions $v_{1(j)}(z, k)$ and $v_{2(j)}(z, k)$ have been found for the j:th layer, the general solution can be written as follows:

$$v_{(j)}(z, k) = c_{1(j)}\, v_{1(j)}(z, k) + c_{2(j)}\, v_{2(j)}(z, k) \tag{4}$$

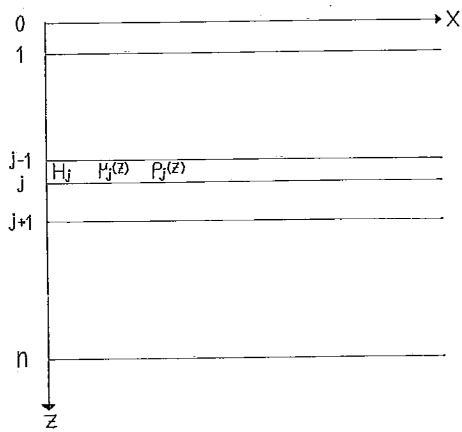

Fig. 1.

For the tension in the layer we get

$$\mu_{(j)}(z)\frac{dv(z\,,k)}{dz} = c_{1(j)}\,\mu_j(z)\,\frac{dv_{1(j)}(z\,,k)}{dz} + c_{2(j)}\mu_j(z)\,\frac{dv_{2(j)}(z\,,k)}{dz} \tag{5}$$

(4) and (5) can be represented as a matrix equation

$$\pi_j(z\,,k) = B_j(z\,,k)\,c_j \tag{6}$$

where the column vector $\pi_j(z\,,k)$ is

$$\pi_j(z\,,k) = \begin{pmatrix} v_j(z\,,k) \\ \mu_j(z)\,\dfrac{dv_j(z\,,k)}{dz} \end{pmatrix} \tag{7}$$

The matrix $B_j(z\,,k)$ has the elements

$$B_{11}(z_{(j)}\,,k) = v_{1(j)}(z\,,k)\,;\quad B_{12(j)}(z\,,k) = v_{2(j)}(z\,,k)$$

$$B_{21(j)}(z\,,k) = \mu_j(z)\,\frac{dv_{1(j)}(z\,,k)}{dz} \text{ and } B_{22(j)}(z\,,k) = \mu_j(z)\,\frac{dv_{2(j)}(z\,,k)}{dz} \tag{8}$$

And the vector

$$c_j = \begin{pmatrix} c_{1(j)} \\ c_{2(j)} \end{pmatrix} \tag{9}$$

The boundary values (2) and (3) are also expressed in vector form as follows:

$$\pi_1(o\,,k) = \begin{pmatrix} v_1(o\,,k) \\ 0 \end{pmatrix} \tag{10}$$

$$\pi_{j-1}(z\,,k) = \pi_j(z\,,k)\,;\quad \text{on the } (j-1)\text{:th boundary, and} \tag{11}$$

$$c_{n+1} = \begin{pmatrix} 0 \\ c_{2(n+1)} \end{pmatrix} \tag{12}$$

If $v_{1(n+1)}(z\,,k) \to \infty$ and $v_{2(n+1)}(z\,,k) \to 0$, when $z \to \infty$, then following Thomson, the origin is placed on the $(j-1)$:th boundary, with z-axes pointing downwards and x-axes to the right. Thus equation (6) is valid on the $(j-1)$:th interface setting $z = 0$ and on the (j):th interface setting $z = H_j$, where H_j is the thickness of the (j):th layer. The result is represented as follows:

$$\pi_j(o\,,k) = B_j(o\,,k)\,c_j \quad \text{and} \tag{13}$$

$$\pi_j(H_j\,,k) = B_j(H_j\,,k)\,c_j \tag{14}$$

From (13) and (14) c_j is eliminated by multiplying (13) from the left by $B_j^{-1}(o\,,k)$ and substituting the product in (14), and the following result is obtained:

$$\pi_j(H_j\,,k) = B_j(H_j\,,k)\,B_j^{-1}(o\,,k)\,\pi_j(o\,,k) \tag{15}$$

Using (15) recursively and the boundary values (11), the relation between $r_{n+1}(o\,,k)$ and $r_1(o\,,k)$ is found to be

$$\pi_{n+1}(o\,,k) = E_n\,E_{n-1}\,\ldots\ldots\ldots\,E_2\,E_1\,r_1(o,\;k) \tag{16}$$

where $E_j = B_j(H_j\,,k)\,B_j^{-1}(o\,,k)$.
From (16) and $\pi_{n+1}(o\,,k) = B_{n+1}(o\,,k)\,c_{n+1}$ the following relation between c_{n+1} and $\pi_1(o\,,k)$ will be obtained:

$$B_{n+1}(o\,,k)\,c_{n+1} = A\,\pi_1(o\,,k) \tag{17}$$

where $A = E_n\,E_{n+1}\,\ldots\ldots\,E_2\,E_1$

or in component form

$$c_{2(n+1)}\,B_{12(n+1)}\,(o\,,k) = A_{11}\,v_1(o\,,k) \quad \text{and}$$

$$c_{2(n+1)}\,B_{22(n+1)}(o\,,k) = A_{21}\,v_1(o\,,k) \tag{18}$$

and dividing both sides of the equations (18), thus eliminating $c_{2(n+1)}$ and $v(o\,,k)$, we get, taking into account (8), the Love wave dispersion relation

$$A_{21} = \frac{\mu(o)}{v_{2(n+1)}(o\,,k)}\,\frac{d\,v_{2(n+1)}(o\,,k)}{dz}\,A_{11} \tag{19}$$

The above method can be applied to the case where $\mu(z)$ and $\varrho(z)$ are linear functions of depth in the j:th layer

$$\mu_j(z) = \mu_{(j-1)} + a_j(z - z_{(j-1)}) \text{ and}$$

$$\varrho_j(z) = \varrho_{(j-1)} + b_j(z - z_{(j-1)}) \tag{20}$$

where $\pi_{(j-1)}$ (and $\varrho_{(j-1)}$) are the values of $\mu_j(z)$ (and $\varrho_j(z)$) on the $(j-1)$:th interface, a_j (and b_j) are the values of the direction coefficients of $\mu_j(z)$ (and $\varrho_j(z)$) in the j:th layer and $z_{(j-1)}$ is the depth of the $(j-1)$:th interface.

The differential equation (1) in the j:th layer will have the form

$$(\mu_{j-1} + a_j(z - z_{j-1})\frac{d^2v}{dz^2} + a_j\frac{dv}{dz} + k^2(c^2\varrho_{j-1} - \mu_{j-1}) + \\ (c^2b_j - a_j)(z - z_{j-1})v = 0 \tag{21}$$

According to SLATER [2], the solution of (21) is in confluent hypergeometric functions

$$v_j(z, k) = e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}z}[c_{1(j)}\Phi(d, 1, az + b) + c_{2(j)}\Psi(d, 1, az + b)] \tag{22}$$

$$\text{where } d = 1/2 - \frac{k^2c^2\mu_{j-1}}{2a_j}\frac{\frac{\varrho_{j-1}}{\mu_{j-1}} - \frac{b_j}{a_j}}{\sqrt{1 - c^2\frac{b_j}{a_j}}} \text{ and}$$

$$az + b = \frac{2k\sqrt{1 - c^2\frac{b_j}{a_j}}}{a_j}[\mu_{j-1} + a_j(z - z_{j-1})]$$

The stress in the j:th layer will have the form

$$\frac{dv_j}{dz} = (\mu_{j-1} + a_j(z - z_{j-1}))e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}z}\left[c_{1(j)}\left(-k\sqrt{1 - c^2\frac{b_j}{a_j}}\Phi(d, 1, az + b) + \right.\right.$$
$$\left. \left. 1\,\Phi(d + 1, 2, az + b)\right) + c_{2(j)}\left(-k\sqrt{1 - c^2\frac{b_j}{a_j}}\Psi(d, 1, az + b) - ad\,\Psi(d + \right.\right.$$
$$\left.\left., 2, az + b)\right)\right] \tag{23}$$

From (22) and (23) the following elements of the matrices $B_j(o, k)$ and $B_j(H_j, k)$ will be obtained:

$$B_{11(j)}(o, k) = \Phi(d, 1, b)$$

$$B_{12(j)}(o, k) = \Phi(d, 1, b) \tag{24}$$

$$B_{21(j)}(o, k) = (\mu_{j-1} - a_jz_{j-1}) - k\sqrt{1 - c^2\frac{b_j}{a_j}}\Phi(d, 1, b) + ad\,\Phi(d + 1, 2, b)$$

$$B_{22(j)}(o,k) = (\mu_{j-1} - a_j z_{j-1}) - k\sqrt{1 - c^2\frac{b_j}{a_j}}\,\Psi(d,1,b) - ad\,\Psi(d+1,2,b)$$

and

$$B_{11(j)}(H_j\,,\,k) \;=\; e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}H_j}\;\Phi\,(d\,,\,1\,,\,aH_j + b)$$

$$B_{12(j)}(H_j\,,\,k) \;=\; e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}H_j}\;\Phi\,(d\,,\,1\,,\,aH_j + b)$$

(25)

$$B_{21(j)}(H_j\,,\,k) \;=\; (\mu_{j-1} + a_j(H_j - z_{j-1})\,)e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}H_j}\left[-k\sqrt{1 - c^2\frac{b_j}{a_j}}\;\Phi\,(d\,,\,1\,,\,aH_j + b) + ad\;\Phi\,(d+1\,,\,2\,,\,aH_j + b)\right]$$

$$B_{22(j)}(H_j\,,\,k) \;=\; (\mu_{j-1} + a_j(H_j - z_{j-1})\,)e^{-k\sqrt{1-c^2\frac{b_j}{a_j}}H_j}\left[-k\sqrt{1 - c^2\frac{b_j}{a_j}}\;\Psi\,(d\,,\,1\,,\,aH_j + b) - ad\;\Psi\,(d+1,\,2\,,\,aH_j + b)\right]$$

Because the expressions (24) and (25) have a rather complicated structure, the layer matrix will not be given explicitly but instead a procedure which can be programmed for a digital computer will be sketched out. The elements of $B_j^{-1}(o\,,\,k)$ are the following:

$$B^{-1}_{11(j)}(o\,,\,k) \;=\; \frac{B_{22(j)}(o\,,\,k)}{\det\,B_j(o\,,\,k)}\,;\qquad B^{-1}_{12(j)}(o\,,\,k) = -\frac{B_{12(j)}(o\,,\,k)}{\det B_j(o\,,\,k)} \qquad (26)$$

$$B^{-1}_{21(j)}(o\,,\,k) \;=\; -\frac{B_{21(j)}(o\,,\,k)}{\det\,B_j(o\,,\,k)} \text{ and } B^{-1}_{22(j)}(o\,,\,k) = \frac{B_{11(j)}(o\,,\,k)}{\det B_j(o\,,\,k)}$$

where $\det\,B_j(o\,,\,k) = B_{11(j)}(o\,,\,k)\;\;B_{22(j)}(o\,,\,k)\;\;B_{12(j)}(o\,,\,k)\;\;B_{21(j)}(o\,,\,k)$ and the elements of E_j are

$$\begin{aligned}
E_{11(j)} &= B_{11(j)}(H_j\,,k)\,B^{-1}_{11(j)}(o\,,\,k) + B_{12(j)}(H_j\,,\,k)\,B^{-1}_{21(j)}(o\,,\,k)\\
E_{12(j)} &= B_{11(j)}(H_j\,,\,k)\,B^{-1}_{12(j)}(o\,,\,k) + B_{12(j)}(H_j\,,\,k)\,B^{-1}_{22(j)}(o\,,\,k) \qquad (27)\\
E_{21(j)} &= B_{21(j)}(H_j\,,\,k)\,B^{-1}_{11(j)}(o\,,\,k) + B_{22(j)}(H_j\,,\,k)\,B^{-1}_{22(j)}(o\,,\,k)\\
E_{22(j)} &= B_{21(j)}(H_j\,,\,k)\,B^{-1}_{12(j)}(o\,,\,k) + B_{22(j)}(H_j\,,\,k)\,B^{-1}_{22(j)}(o\,,\,k)
\end{aligned}$$

In this connection, two remarks may be made concerning the above precedure. Firstly, when d is complex and $az + b$ is imaginary $\left(\text{that is, when } c < \sqrt{\frac{a_j}{b_j}}\right)$, the solution $v_j(z, k)$ will also be complex and we shall have to develop a subroutine to calculate the product of the two complex numbers. Secondly, when $j = n + 1$ (the halfspace) and $c < \sqrt{\frac{a_{n+1}}{b_{n+1}}}$, the asymptotic behavior of the solutions will be

$$\left| e^{-k\sqrt{1-c^2\frac{b_{n+1}}{a_{n+1}}}\,z}\,\Phi\,(d\,,1\,,az+b)\right| \to \infty\,,\quad \text{when} \quad z \to \infty\,, \text{ and} \tag{28}$$

$$\left| e^{-k\sqrt{1-c^2\frac{b_{n+1}}{a_{n+1}}}\,z}\,\Psi\,(d\,,1\,,az+b)\right| \to 0\,,\quad \text{when} \quad z \to \infty$$

In the second case, the method is applied to the case where $\mu(z)$ and $\varrho(z)$ are constants μ_j and ϱ_j in the j:th layer (the original Thomson—Haskell case). The differential equation (1) will be

$$\frac{d^2v}{dz^2} + k^2\left(\frac{c^2}{\beta_j^2} - 1\right)v = 0 \tag{29}$$

where $\beta_j = \sqrt{\frac{\mu_j}{\varrho_j}}$

The solution of (29) will be

$$v_j(z\,,k) = c_{1(j)}\,e^{ik\pi\beta_j z} + c_{2(j)}\,e^{-ik\pi\beta_j z} \tag{30}$$

$$\text{where} \quad \pi_{\beta j} = \begin{cases} \sqrt{\frac{c^2}{\beta_j^2} - 1}\,, \text{ when } c < \beta_j \\ i\sqrt{1 - \frac{c^2}{\beta_j^2}}\,, \text{ when } c < \beta_j\,. \end{cases}$$

The tension in the j:th layer will be

$$\mu_j \frac{dv_j(z\,,k)}{dz} = i\,k\,\pi_{\beta j}\,\mu_j\,c_{1(j)}\,e^{ik\pi\beta_j z} - ik\,\pi_{\beta j}\,\mu_j\,c_{2(j)}\,e^{-ik\pi\beta_j z} \tag{31}$$

From (30) and (31) the matrices $B_j(o\,,k)$ and $B_j(H_j\,,k)$ will be

$$B_j^{-1}(o\,,k) = \frac{1}{2}\begin{pmatrix} 1 & \dfrac{1}{ik\,\pi_{\beta j}\,\mu_j} \\ 1 & -\dfrac{1}{ik\,\pi_{\beta j}\,\mu_j} \end{pmatrix} \tag{32}$$

and

$$B_j(H_j\,,k) = \begin{pmatrix} e^{ik\,\pi_{\beta j}H_j} & e^{-ik\,\pi_{\beta j}H_j} \\ ik\,\pi_{\beta j}\,\mu_j\,e^{ik\,\pi_{\beta j}H_j} & -\,ik\,\pi_{j\beta}\,\mu_j\,e^{-ik\,\pi_{\beta j}H_j} \end{pmatrix} \tag{33}$$

The matrix $E_j = B_j(H_j\,,k)\,B_j^{-1}(o\,,k)$ has two different forms, according whether $\pi_{\beta j}$ is real or imaginary. Namely, when $\pi_{\beta j}$ is real

$$E_j = \begin{pmatrix} \cos\,k\pi_{\beta j}\,H_j & \dfrac{\sin\,k\,\pi_{\beta j}\,H_j}{k\pi_{\beta j}\,\mu_j} \\ -\,k\pi_{\beta j}\,\mu_j\,\sin\,k\,\pi_{\beta j}\,H_j & \cos\,k\,\pi_{\beta j}\,H_j \end{pmatrix} \tag{34}$$

and when π_{gj} is imaginary

$$E_j = \begin{pmatrix} \cosh\,k\,\pi_{\beta j}\,H_j & \dfrac{-\,\sinh\,k\,\pi_{\beta j}\,H_j}{k\,\pi_{\beta j}\,\pi_j} \\ k\,\pi_{\beta j}\,\mu_j\,\sinh\,k\,\pi_{\beta j}\,H_j & \cosh\,k\,\pi_{\beta j}\,H_j \end{pmatrix} \tag{34'}$$

Now, it is possible to express the terms of the dispersion relation (19) explicitly, when $\mu(z)$ and $\varrho(z)$ in the layers and in the halfspace are either linear functions or constants. Especially when in the half-space, $\mu(z)$ and $\varrho(z)$ are linear functions

$$\mu_{n+1}(z) = \mu_n + a_{n+1}(z - z_n) \quad \text{and} \quad \varrho_{n+1}(z) = \varrho_n + b_{n+1}(z - z_n)\,.$$

From (20) and (25) we get

$$\frac{A_{21}}{A_{11}} = \left[\frac{k^2c^2\mu_n}{a_{n+1}}\left(\frac{\varrho_n}{\mu_n} - \frac{b_{n+1}}{a_{n+1}}\right) - k\sqrt{1 - c^2\,\frac{b_{n+1}}{a_{n+1}}}\right]\,\frac{\Psi(d+1\,,2\,,b)}{\Psi(d\,,1\,,b)} - \\ - k\sqrt{1 - c^2\,\frac{b_{n+1}}{a_{n+1}}}\,. \tag{35}$$

Conclusions:

1. According to the above method it is possible to fit $\mu(z)$ and $\varrho(z)$ more accurately to the real situation. Especially, the influence of the deep structure and the regions where $\mu(z)$ and $\varrho(z)$ change rapidly can be taken into account better.

2. A drawback of the method is the labour involved in the calculation of special functions. But that ought to be remedied by the use of a fast digital computer.

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