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Low Order Baroclinic Models Forced by Meridional and Zonal Heating

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Abstract

The low-order Lorenz model for interannual variability is investigated. It is found that the chaotic behavior is due to the neglect of the beta effect in the basis equations. If the beta effect is included the solutions become periodic.

A six component baroclinic model containing meridional heat transport, but no momentum transport, heated in the meridional as well as the zonal plane is analysed. Steady state solutions and their stability are determined. For sufficiently low meridional forcing we find only stable solutions. Larger meridional forcing introduces instability provided the zonal forcing by heating is not too large. For large meridional and zonal heating multiple steady state solutions exist, but one of these is always stable. This stable solution has the lowest value of the vertical wind shear. Numerical integrations of the nonlinear equations indicate that the unstable solutions are periodic. The statements above apply to a single wavelength of 5500 km, where one finds that the resulting waves transport sensible heat from south to north.

The last section of the paper contains an analysis of the model behavior, when the wavelength is increased to, say 10000 km. While stable steady state solutions exist also in this case, it is characteristic of all such solutions that they transport sensible heat from north to south contrary to the observed quasi-stationary waves. A comparison is made with the classical linear studies of the maintenance of the long permanent waves, and it is shown that the apparent success of these studies is due to the selection of a rather small meridional scale.

It is demonstrated that the Lorenz model is a special case of the standard two level, quasi-geostrophic model. The Lorenz-model appears, if assumptions are made relating the parameters of the vertical mean flow to those of the thermal flow. The assumptions regarding the wave amplitudes result in a baroclinic wave, where the thermal flow is lagging the vertical mean flow by a quarter of a wavelength. The zonal wind in the vertical mean flow has to be related to the zonal wind of the thermal flow. In the model used in this study the relation between the two quantities is obtained from the requirement that the zonal flow is in equilibrium from an energetical point of view.

1. *Introduction*

The processes leading to the observed interannual variability in the atmosphere have not yet been fully apprehended. The main problem is to explain in which way we obtain a non-periodic response to a primary forcing, which is essentially periodic, since we may assume with good accuracy that the local diabatic heating due to radiation has a high

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amplitude period of one year. One possibility is that the interannual variability may be explained as an internal dynamical process. If this is not the case, we are left with the problem of describing the external processes, which may account for the nature of interannual variability. A candidate is the El Nio- Southern Oscillation processes, which would link the interannual variability to the interactions between the atmosphere and the oceans.

Lorenz (1984 and 1990) has argued that an internal mechanism may be sufficient to account for interannual variations. The main argument is that the response to the external forcing in terms of the differential heating between Equator and Pole in an hemisphere is periodic and therefore non-chaotic as long as the differential heating is below a critical value, while the response is chaotic, when the differential heating exceeds the critical value. Since the heating difference seems to be large enough in the winter season and subcritical during the summer, we should conclude that the atmospheric behavior is chaotic in the cold and non-chaotic in the warm season. These conclusions are reached after an analysis of a simple general circulation model having only three orthogonal components. One component describes the vertical wind shear, while the other two are the amplitudes of the sine and cosine components of a travelling wave. It is not a straightforward matter to derive the equations for such a model, but it can be considered as a special case of a six component version of the classical two level, quasi-geostrophic model. *Lorenz* (1990) says that his model equations were obtained in a somewhat ad hoc manner. In any case, the equations describe supposedly the development of the thermal flow in such a model using only three components. To obtain the equations in a closed form it is necessary to relate the zonal component in the vertical mean flow to that of the vertical shear flow and to relate the wave in the vertical mean flow to the wave in the vertical shear flow.

No unique way exists to satisfy these requirements, but it is clear from the equations that the thermal wave is lagging the wave in the mean flow by exactly one quarter of a wavelength in the Lorenz model. A typical wave in the atmosphere behaves differently, because both the quasi-geostrophic theory and observations agree that a baroclinic wave may have a large phase difference initially between the two fields, but the phase lag decreases to smaller values as the wave develops and matures without ever reaching the state of an equivalent barotropic structure. The relation between the zonal windshear in the thermal flow and the zonal wind in the vertical mean flow is unstated for the Lorenz model. Such a relation could be as simple as a proportionality factor, but we shall discuss and investigate a different approach in the second section of this paper. However, the assumption should be considered together with the formulation of the boundary layer dissipation. We note finally that the Lorenz model does not contain a beta effect.

In an earlier paper (*Wiin-Nielsen*, 1990) the author analysed the response of the six component model to external forcing which, however, was restricted to differential heating in the south-north direction only. In that case it was found that to each intensity of the heating there existed one and only one stable stationary state. These results are not directly comparable to the Lorenz model, because he includes a differential heating in the west-east direction as well, but a series of long term integrations of the six component model equations with a heating equivalent to the one in the Lorenz model has been carried out. The results (*Wiin-Nielsen*, 1991) indicate that all the numerical integrations lead to solutions which are periodic in time. It has thus been impossible to find chaotic states in this slightly more general model. These results are in agreement with those obtained by *de Swart* (1988), who finds that a minimum number of components of eight are necessary to obtain chaotic solutions in a two level, quasi-geostrophic model. It would thus appear that the chaos found by Lorenz in his three component model is somewhat artificial and is introduced by the assumptions made to reduce the model to very few components. It is stressed that the investigations reported here will not be able to answer the question: "Can interannual variability be explained as an internal dynamical response?" The answer will in the opinion of the author require investigations of models with a higher number of components. The present understanding is far from clear. From work finished very recently it has been shown (*Kuemmel*, 1991) that several chaotic attractors exist in a twelve component baroclinic model. However, it is also known that chaos in the flow may disappear with increased resolution indicating that the results from low-order models are correct for the model, but not applicable to the atmosphere with its many degrees of freedom. On the other hand, we know that the atmospheric flow is chaotic in the basic sense that predictions made from two initial states which are different, but close to eachother will deviate very significantly from each other after a couple of weeks of integration. It is, however, not known in which way this basic limited predictability of the atmosphere will influence the climatic state derived by averaging from a long term integration of a climate model which in a sense should have "forgotten" its initial state.

One of the purposes of this investigation is to explore why chaos appear in the Lorenz model. This is done by comparing with a modified model having also three components (section 2), while a second purpose is to make an analysis of the six component model from a theoretical and a numerical point of view (section 3). Section 4 contains an analysis of the behavior of the six component baroclinic model, when the wavelength is large. The special case of zonal heating only is described briefly in section 5, while summary and conclusions are found in section 6.

2. *A modified Lorenz model*

Low order models are formulated for certain purposes. When we want to make a model which could qualify as a simple general circulation model of the atmosphere, it may be important that the model contains the major processes in the dynamics of the atmosphere albeit in a rudimentary form. For example, the models of Lorenz and the author, mentioned in the introduction, do not include the meridional transport of momentum by the waves. Since we know that the momentum transport has a marked influence on the structure of the zonal wind, one could argue that it ought to be represented in the model. On the other hand, this conversion from the eddy kinetic to the zonal kinetic energy is a relatively minor contribution to the energetics of the atmosphere. However, while the kinetic energy transfer from the eddies to the zonal current according to observational studies is positive most of the time, it may reverse its sign during periods of major blocking pattern in the Northern Hemisphere as shown by *Wiin-Nielsen et al.* (1964). It appears therefore that it would be worth while to investigate models in which the momentum transport is present. It is, however, known that the minimum number of components in a model satisfying these requirements is twelve, and in that case it appears out of the question to rely on anything but long term numerical integrations.

From an energetical point of view the Lorenz model has generations of zonal and eddy potential energy by the heating imposed on the model. The heat transport by the eddies will result in an energy conversion between the eddies and the zonal baroclinic component (the shear flow or the thermal flow) in a two level model. In the real atmosphere there is also an energy conversion from the baroclinic to the barotropic component (the vertical mean flow). The latter conversion will also be present in the Lorenz model, because the vertical mean flow will have to be related in some fashion to the vertical shear flow. A simple proportionality factor may be too crude, because the structure of the atmosphere may change significantly during the developments. To be specific we reproduce the equations for the six component model. They may be written in the form:

$$\frac{dB_*}{dt} = -e\,(B_* - 2B_T)$$

$$\frac{dB_T}{dt} = -a_0\,(E_*F_T - E_TF_*) + e\,(B_* - 2B_T) - e_TB_T + g_zQ_z$$

$$\frac{dE_*}{dt} = (a_*B_* - b_*)F_* + a_*B_TF_T - e(E_* - 2E_T)$$

$$\frac{dF_*}{dt} = -\,(a_*B_* - b_*)E_* - a_*B_TE_T - e(F_* - 2F_T)$$

$$\frac{dE_T}{dt} = (a_TB_* - b_T)F_T - c_mB_TF_* + e_1(E_* - 2E_T) - e_2E_T + g_eQ_s$$

$$\frac{dF_T}{dt} = -\,(a_TB_* - b_T)E_T + c_mB_TE_* + e_1(F_* - 2F_T) - e_2F_T + g_eQ_c \qquad (1)$$

The dependent variables in (1) are the velocity components B_*, B_T, E_*, F_*, E_T and F_T, and they are in turn defined through the following specification of the streamfunction of the vertical mean flow:

$$\psi_* = \frac{B_*}{2\lambda}\sin(2\lambda y) + \frac{E_*}{k}\sin(\lambda y)\sin(kx) + \frac{F_*}{k}\sin(\lambda y)\cos(kx) \qquad (2)$$

with an analogous expression for the thermal streamfunction. The various coefficients in (1) are defined by the expressions:

$$N = 1 + \frac{q^2}{4\lambda^2}$$

$$a_0 = \frac{q^2}{4kN}$$

$$N_l = k^2 + \lambda^2 + q^2$$

$$N_s = k^2 + \lambda^2$$

$$a_* = k\frac{k^2 - 3\lambda^2}{2(\lambda^2 + k^2)}$$

$$b_* = \frac{\beta k}{\lambda^2 + k^2}$$

$$a_T = k\frac{k^2+q^2-3\lambda^2}{2N_l}$$

$$b_T = \frac{\beta k}{N_l}$$

$$c_m = k\frac{q^2+3\lambda^2-k^2}{2N_l}$$

$$e = \frac{\varepsilon}{N}$$

$$e_T = \frac{\varepsilon_T}{N}$$

$$e_1 = \frac{\varepsilon N_s}{N_l}$$

$$e_2 = \frac{\varepsilon_T N_s}{N_l}$$

$$g_z = \frac{\kappa q^2}{4\lambda f_0 N}$$

$$g_e = \frac{\kappa q^2 k}{2 f_0 N_l} \tag{3}$$

A number of constants enter the problem. The numerical values and definitions are listed below:

$$q^2 = 2.25 \times 10^{-12} m^{-2}$$

$$k = 2\pi/L; L = 5.5 \times 10^6 m$$

$$\lambda = \pi/W; W = 1.0 \times 10^7 m$$

$$f_0 = 1.0 \times 10^{-4} s^{-1}$$

$$\beta = 1.6 \times 10^{-11} m^{-1} s^{-1}$$

$$\kappa = R/c_p = 0.286$$

$$\varepsilon = 2.0 \times 10^{-6} s^{-1}$$

$$\varepsilon_T = 1.2 \times 10^{-6} s^{-1} \tag{4}$$

We shall next look at the problem of obtaining a three component system from the three equations describing the thermal flow. Whenever a quantity relating to the vertical mean flow appears in the system (1), it must be related to the thermal flow. The quantity $(E_* F_T - E_T F_*)$ appearing in the first equation in the system (1) is a measure of the sensible heat transport in the model. We shall follow Lorenz and define $E_* = F_T$ and $F_* = -E_T$, which corresponds to a wave where the thermal flow is lagging the vertical mean flow by one quarter of a wavelength. Other assumptions could be made by adding a factor less than unity to these relations, but it would not change the fact that the thermal flow is "frozen" with a constant phaselag relative to the vertical mean flow.

The other quantity which needs to be replaced is the zonal component B_*. One could for example set $B_* = 2B_T$, but in that case we notice from the equations that the influence of the planetary boundary layer would be eliminated. In the modified Lorenz model we shall proceed to determine B_* in such a way that the vertical mean flow is balanced from an energetical point of view. This means that we equate the energy conversion from the baroclinic flow with the dissipation of the kinetic energy of the vertical mean flow. For this purpose it is convenient to divide the total energy in the model in two parts. One part will be the total energy E_t in the baroclinic flow. It consists of the available potential energy and the kinetic energy of the baroclinic component. The other part is the kinetic energy of the barotropic flow. A calculation of the energy conversion between these two parts in the model leads to the result that

$$C(E_t,K_*) = dB_T(E_*F_T - E_TF_*) \tag{5}$$

where the constant d has the value

$$d = \frac{k^2 - 3\lambda^2}{8k} \tag{6}$$

The dissipation of the kinetic energy of the vertical mean flow is

$$D(K_*) = \frac{\varepsilon}{4}\left(2(B_*^2 - 2B_TB_*) + \frac{N_s(E_T^2 + F_T^2)}{k^2}\right) \tag{7}$$

where the relations between the wave amplitudes have been introduced already. Equating the two expressions in (6) and (7) it is seen that we obtain a quadratic equation from which we may obtain B_* from B_T, E_T and F_T provided real solutions exist to the equation. Such solutions will not exist in all cases, but it is easy to see that they will exist if B_T is sufficiently large. In view of these considerations we have a model in which the zonal component of the vertical mean flow may change during the time integrations, since it can be computed in every time step.

Integrations of the model equations for the modified Lorenz model have been carried out for a large number of cases, where Q_z has been varied from small to large values, while the value of Q_c has been kept constant in each series. The maximum value of Q_z was 3.0×10^{-2} $kJt^{-1}s^{-1}$, which is somewhat larger than the value computed by *Lawniczak* (1970) for the months of January and the results of the study carried out by *Schaack et al.* (1990) on the basis of data from the Global Weather Experiment. It also appears that the selected maximum value is too large for atmospheric conditions because the generation of available potential energy for the maximum case is considerably larger than observational values. For all integrations it can be said that they eventually lead to periodic oscillations. We shall demonstrate a single case, and we have selected $Q_z = 2.0 \times 10^{-2}$ $kJt^{-1}s^{-1}$ and $Q_c = 1.0 \times 10^{-3}$ $kJt^{-1}s^{-1}$. Fig. 1 shows the variation of the two zonal windcomponents, the change in the generation, the conversion and the dissipation of the thermal energy, and the imbalance of these three quantities during a period. The selected period is well into a long term integration after the influence of the initial state has disappeared. We note that the change of the zonal winds is rather small being less than 0.5 $m\ s^{-1}$. The same may be said about the energetical quantities, where the total variation is less than 0.5 Wm^{-2}. The total thermal energy is not in complete balance, but the difference between input and output in the thermal energy goes through a periodic change with an amplitude of almost 0.2 Wm^{-2}.

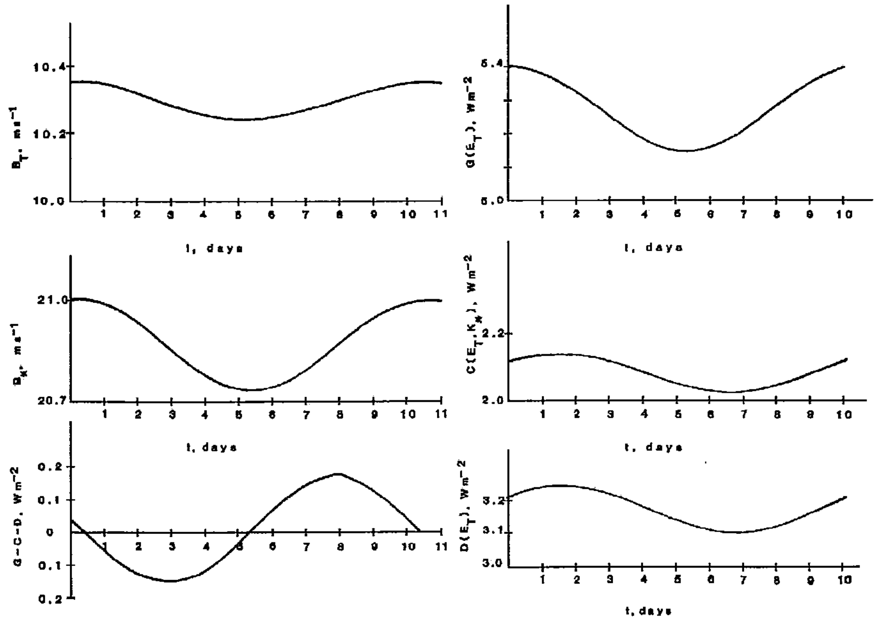

Fig. 1. The periodic behavior of a single integration with $Q_z = 2.0 \times 10^{-3}\, kJ\, t^{-1}\, s^{-1}$ and $Q_c = 1.0 \times 10^{-3}\, kJ\, t^{-1}\, s^{-1}$. The figure contains the two zonal winds as the two upper figures to the left, while the three figures in the right column show the generation, the conversion and the dissipation in the unit $W\, m^{-2}$. The lower left figure is the imbalance of the thermal energy. All figures show the variation over one period.

We may thus conclude that the experiments, which have been performed, do not lead to a chaotic state for the modified Lorenz model, when the intensity of the forcing is kept within a range, which could occur in the atmosphere. An investigation of the steady states of the modified Lorenz model has been done, and the stability of these states has been determined. The main result is that all the steady states are unstable, and that at least one of the eigenvalues of the unstable steady states indicate an oscillatory mode. These results are in agreement with the behavior of the numerical integrations, because they lead to a "limit-cycle" in the threedimensional phase space.

The integrations have been carried out for sufficiently large values of Q_z. We may see this by inspecting Fig. 2, where the generation of the thermal energy is shown as a function of Q_z. The figure shows that the generation for the largest values of the zonal heating is well above any estimate from data studies. In view of the results obtained with the modified Lorenz model showing only periodic solutions, one may ask why it does not contain chaotic solutions. A possibility could be that the special way of calculating the zonal part of the vertical mean wind account for the different behavior of the two models, but this is not the case. It turns out that the quantity B_* for reasonably large values of Q_z is almost twice the value of B_T. An integration of the equations for the modified model with $B_z = 2\, B_T$ (i.e. as in the Lorenz model) gives also only periodic solutions. We may thus conclude that the only difference between the two models is the inclusion of the beta effect in the modified model, while Lorenz did not include it in his three component model. When the modified model is brought into the form of the Lorenz model, we get a result as shown in Fig. 3, where the integration was carried out with $Q_z = 3.0 \times 10^{-2}\ kJt^{-1}\ s^{-1}$. Fig. 3 shows that the solution has become irregular and non-periodic, although it appears to have a periodic envelope. In Fig. 3 we have plotted the generation of thermal energy as a function of time, but only the values which are two hours apart has been included. The same information is given in Fig. 4, where we have plotted the maximum and minimum values. The extrema are situated with a time interval of about 3.375 hours.

The main conclusion is therefore that the well known stabilizing effects of the beta terms are responsible for the periodic behavior of the modified model. We may investigate the typical time scales involved in the modified model by considering the zonal windcomponents as constant, neglecting the forcing and dissipation and thereby linearizing the problem. We consider therefore the last two equations in the system (1). It should be noted that the terms containing the factor e_1 and normally considered as part of the frictional terms actually work as a reduction of the beta effect in the modified model, and these terms will therefore be included in the calculations. With these remarks we may write the linearized equations in the following form:

$$\frac{dW}{dt} = (\alpha_2 - i\alpha_1)W \tag{8}$$

where $W = E_T + i\,F_T$, and where the coefficients are defined by

$$\alpha_1 = a_T B_* - b_T + e_1$$

$$\alpha_2 = c_m B_T \tag{9}$$

The solution of (8) is straightforward. α_1 determines the frequency or the period of the oscillation. Using the same constants as before and setting $B_* = 22\ m\ s^{-1}$ taken as a typical value from the integrations with rather large values of Q_z we find that the period become 10.4 days. α_2 is a measure of the growth rate. With $B_T = 11\ m\ s^{-1}$ we find an e-folding time of 5.4 days. These values are in good agreement with the actual integrations of the nonlinear equations. On the other hand, if we disregard those terms, which are not included in the Lorenz model, the corresponding period of oscillation is 6.5 days, while the e-folding time becomes the same, but these numbers are rather meaningless because the zonal winds varies a great deal in the *Lorenz* integrations as seen from his paper (1990).

The comparative study indicates that the chaotic behavior of the original Lorenz model is due to the neglect of the beta-terms in the equations. It would thus appear that the model produces a chaos, which we are unlikely to observe in the atmosphere.

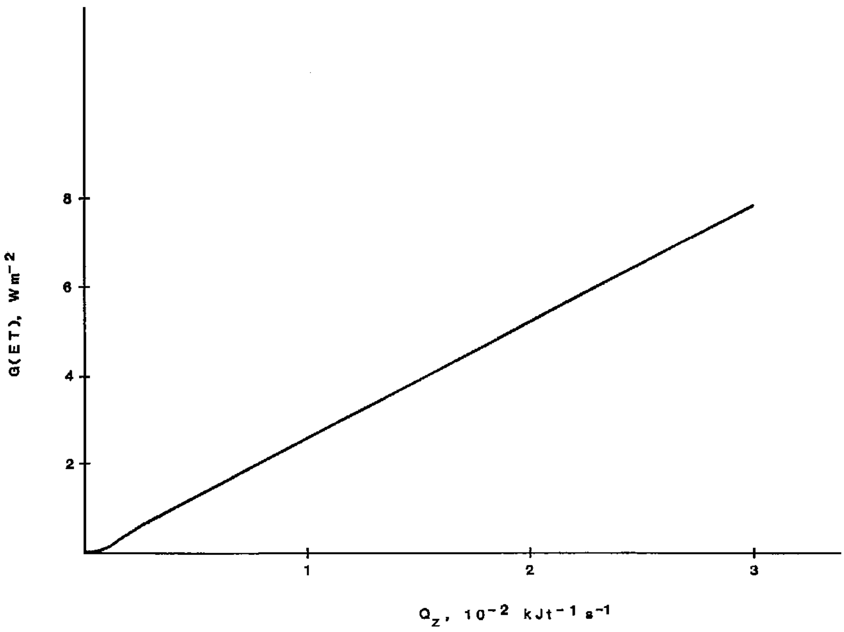

Fig. 2. The generation of the thermal energy as a function of the meridional heating.

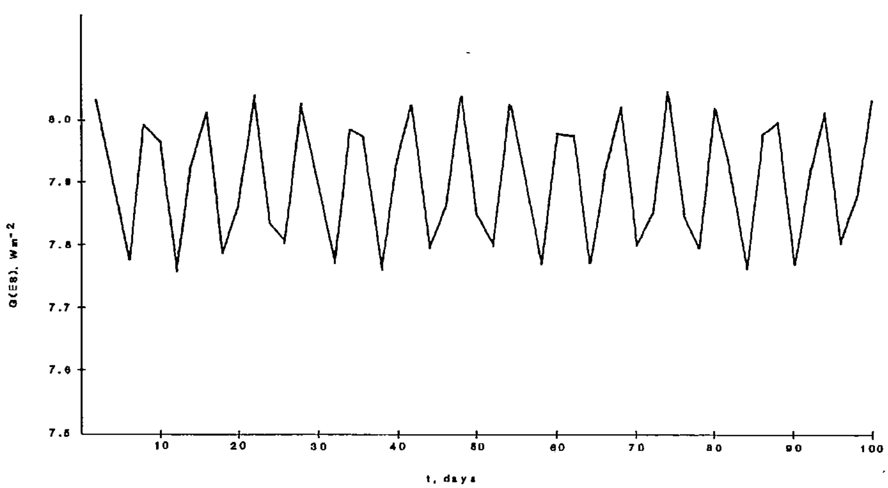

Fig. 3. The generation of the thermal energy for a one hundred day interval well into a long term integration for $Q_z = 3.0 \times 10^{-2}\, kJ\, t^{-1}\, s^{-1}$. Data points are plotted with an interval of 2 days. Beta-effect is not included.

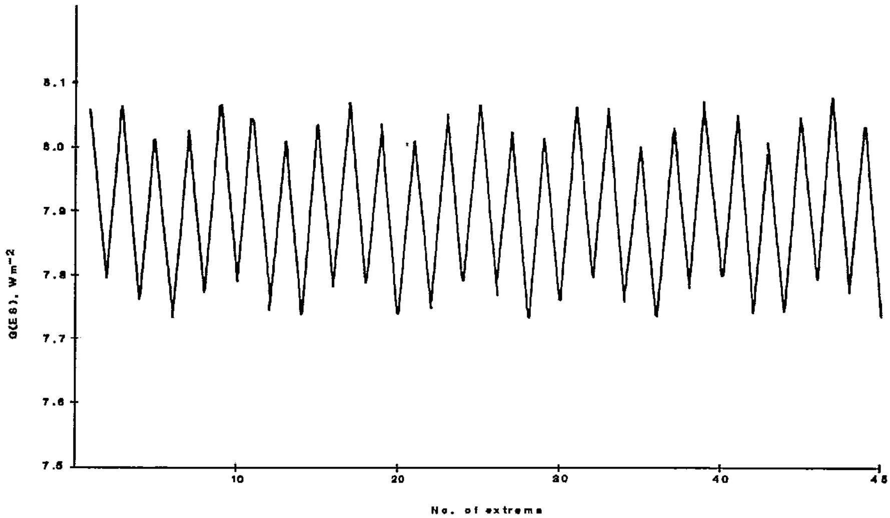

Fig. 4. The extreme values of the generation plotted as a function of the number of extrema in the record. The period is approximately 3.75 days. Beta-effect is not included.

3. *The six component model*

In the present section we shall present a brief analysis of the six component model. In the case of zonal forcing only the model has already been treated by the author (1990) with the result that a stable steady state exists for each value of the zonal forcing. With no heating in the west-east direction it turns out that the four equations for the eddy components can be replaced by another set of equations expressing the rate of change of quantities which are proportional to 1) the heat transport by the waves, 2) the kinetic energies of the thermal flow and 3) the vertical mean flow and 4) another transport quantity which is proportional to the transport $(v_* v_T)_z$, where v is the meridional wind component and the subscipt z refers to a zonal average. The inclusion of the heating in the west-east direction prevents the use of the technique mentioned above. The more general six component model is therefore more cumbersome to deal with.

The full system of equations for the six component model is the system (1). We note that in the present model we have adopted the assumption that the quantities at the lowest level can be obtained by linear extrapolation from above. These six equations may be integrated in time by selecting a time-differencing scheme. Such integrations using the Heun scheme will be presented later in this section. As this scheme may be unstable for too large time steps, we have repeated some of the integrations with lower timesteps to be sure that no instability has occured.

The steady states of the model may be obtained by observing first that a steady state requires that $B_* = 2\,B_T$ as seen from the first equation in (1), which is very simple in the present model, because then the selected waves do not transport eddy momentum. The next step is to set the four time-derivatives equal to zero in the equations for the development of the eddy quantities. These four equations are linear in E_*, F_*, E_T and F_T and may be solved for these quantities in terms of B_T, Q_z and Q_c. To ease the calculations it is an advantage to introduce the following notations:

$$P_0 = a_* B_T \qquad P_1 = 2\,a_* B_T - b_* \qquad P_2 = P_0 + 2P_1$$

$$R_0 = c_T \qquad R_1 = 2\,a_T B_T - b_T \qquad R_2 = 2R_0 + R_1 \tag{10}$$

After considerable calculations it turns out that the four amplitudes of the wave components can be written in the forms:

$$E_* = g_e Q_c(2e_0^2 R_2 + R_0 P_0 P_2 + e_0 e_3 P_2 - R_2 P_0 P_1)/D$$

$$F_* = -g_e Q_c(e_0 R_1 P_2 + e_2 P_0 P_1 + e_1 P_0 P_2 + 2e_0^2 e_2)/D$$

$$E_T = g_e Q_c(e_0^2 R_2 - R_0 P_1 P_2 + e_0 e_1 P_2 + R_2 P_1^2)/D$$

$$F_T = g_e Q_c(e_0 R_0 P_2 + e_2 P_1^2 + e_1 P_1 P_2 + e_0^2 e_2)/D \tag{11}$$

where D is defined by

$$D = (R_0 P_2 - R_2 P_1)^2 + (e_1 P_2 + e_2 P_1)^2 + e_0^2(e_2^2 + R_2^2) + 2e_0 P_2(e_2 R_0 + e_1 R_2) \tag{12}$$

The next task is to compute the quantity, which is a measure of the heat transport, i.e. T = $E_* F_T - E_T F_*$. We get

$$T = \frac{g_e^2 Q_c^2}{D} e_0 P_2 \tag{13}$$

The final step is to insert the expression for T in the second equation of the system (2.1) with the result that

$$Q_c^2 = \frac{D}{a_0 e_0 g_e^2 P_2}(g_z Q_z - e_T B_T) \tag{14}$$

Equation (14) is formally an equation of the fifth degree, but we are interested only in the real roots of the equation. These may be visualized graphically by calculating Q_c as a function of B_T for selected values of Q_z. In this regard we notice that the left hand side of the equation is positive. Since it can be shown that D is positive in the area of interest, it follows that in the region, where P_2 is positive, i.e.

$$B_T > \frac{2b_*}{5a_*} \tag{15}$$

we must also require that the other factor on the right hand side of (14) is positive giving the condition

$$B_T < \frac{\kappa}{f_0} \frac{q^2}{4\lambda\varepsilon_T} Q_z \tag{16}$$

On the other hand, if P_2 is negative we should reverse the inequality signs in (15) and (16). With the value which we have used for the wavelength (5500 km) we find that (15) and (16) in numerical terms may be combined to

$$12.735 < B_T < 4.2673\hat{Q}_z \tag{17}$$

where the unit of the heating term is $10^{-3}\ kJ\ t^{-1}\ s^{-1}$. From (17) it is seen that the two cases are separated by $\hat{Q}_z = 3$.

Fig. 5 shows two examples of the graphical representation for small values of the zonal heating ($\hat{Q}_z = 1$ for the left figure and 2 for the right figure). The asymptote for the larger values of the vertical wind shear represents the position, where P_2 becomes zero. We recall that the curves really are symmetric around the horizontal axis, because both + and - are present, when we solve for the west-east heating. The curves are monotonic giving a single solution for each value of the west-east heating.

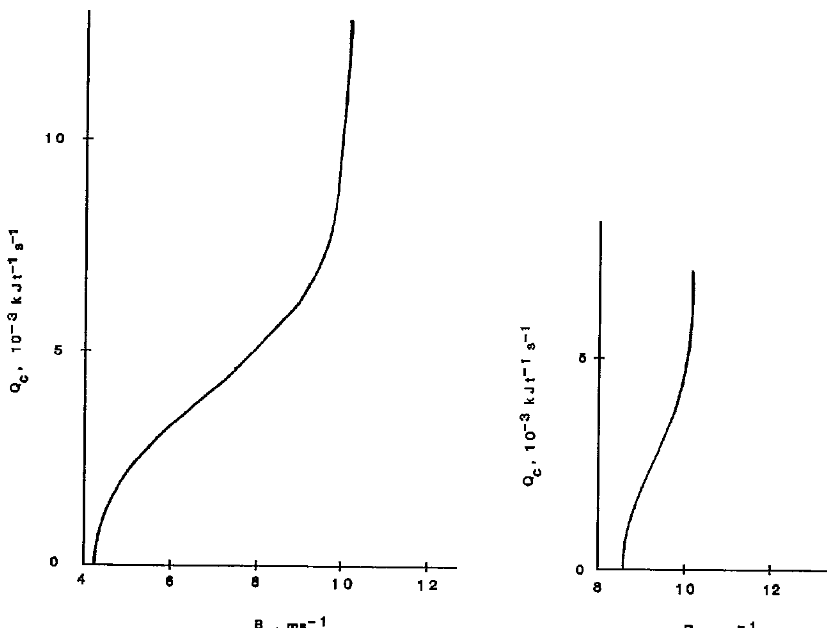

Fig. 5. The zonal heating as a function of the zonal windshear under steady state conditions. $Q_z = 1.0 \times 10^{-3}$ $kJ\,t^{-1}\,s^{-1}$ for the left figure, while it is twice as large for the right figure. Note that only one steady state value is found in these cases.

In Fig. 6 we show an example for a moderate value of the zonal heating ($\hat{Q}_z = 4$). The curve has the asymptote to the left for the smaller values of the vertical wind shear. It has also a minimum and a maximum indicating that three solutions exist within a certain interval of the west-east heating. In this regard the curve is very similar to the analogous curve for the Lorenz model. We note that the triple solutions can be obtained for moderate values of the west-east heating. Fig. 7 displays a more extreme case, where the zonal heating is rather large ($\hat{Q}_z = 10$). The curve has the same characteristics as in Fig. 6, but the scale has been changed in such a way that it is now 10 times larger. Three solutions will be present except when the west-east heating is rather small. One of these will in the present examples have a value slightly larger than 10 $m\,s^{-1}$, while the other solutions will have a much larger value of the vertical wind shear. In the case of Fig. 7 the largest solution would be out of the normal atmospheric range.

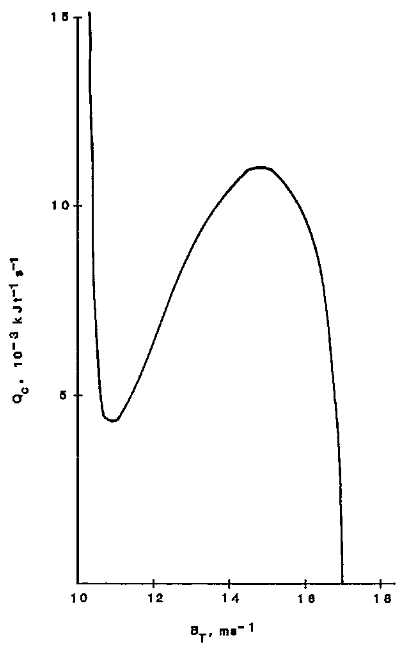

Fig. 6. Similar to Fig. 5, but with $Q_z = 4.0 \times 10^{-3}\,kJ\,t^{-1}\,s^{-1}$. Note, that three steady states exist for a sufficiently large value of Q_c.

A numerical solution of the equation for the steady state values of the vertical wind shear is naturally also possible. Such solutions have been obtained in various cases. Fig. 8 shows an example for $Q_z = 5 \times 10^{-3} kJt^{-1}s^{-1}$. The steady state value of the vertical wind shear is given as a function of the west-east heating. For a number of cases we have determined the stability of the steady state. The sections containing unstable solutions are marked with the letter: U, while stable sections are marked by: S. Fig. 8 shows thus that the large and the intermediate values of the vertical wind shear represent unstable solutions, while the small values are stable solutions. The same pattern can be seen in Fig. 9, which is drawn for the larger value $Q_z = 8.0 \times 10^{-3}$. Fig. 10 contains a general view of the steady states and their stability as a function of the south-north and west-east heating. The diagram shows three regions: A lower region for small values of the south-north heating, where stable solutions exists, an intermediate region, where each point has three stationary states, of which two are unstable and one is stable, and finally a region of instability for large values of the south-north heating. The west-east heating has a stabilizing effect in such a way that sufficiently large values of it will change an unstable solution to a stable one. The main purpose of these figures is to illustrate the properties of the model, although it is realized that the parameter space goes beyond the observed values.

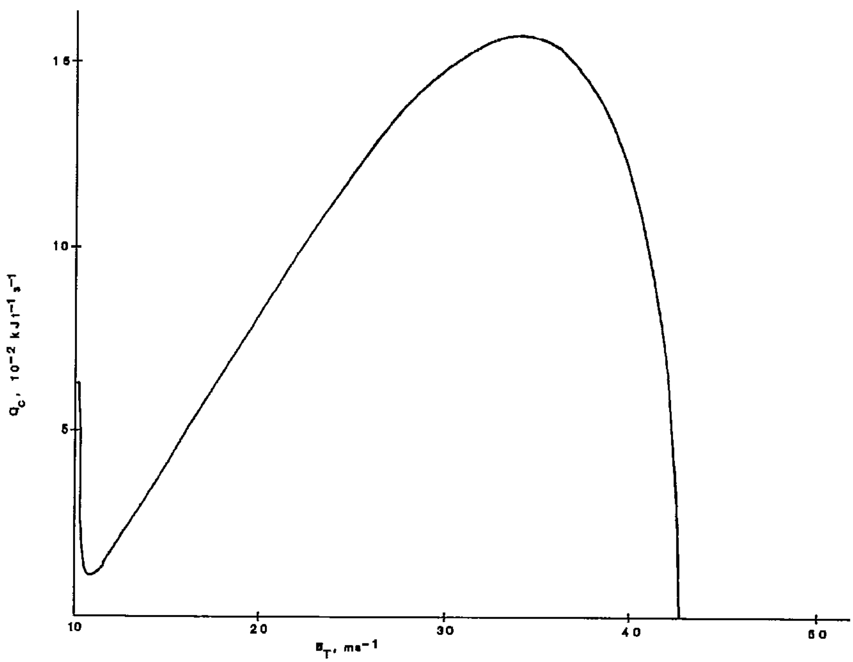

Fig. 7. Similar to Fig. 6, but with $Q_z = 1.0 \times 10^{-2}\, kJ\, t^{-1}\, s^{-1}$.

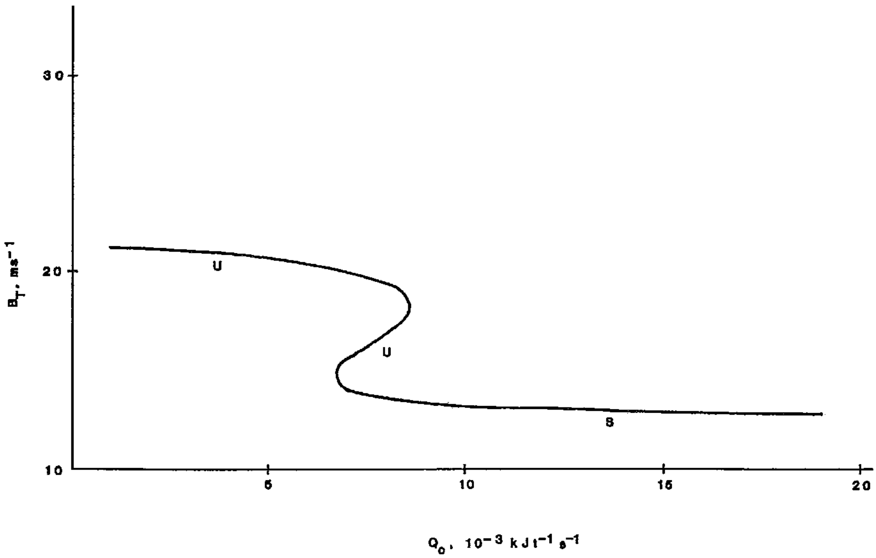

Fig. 8. The steady state values of the zonal vertical windshear as a function of the zonal heating for Q_z = 5.0×10^{-3} $kJ\,t^{-1}\,s^{-1}$. The letters U and S on the various parts of the curve indicate the stability of the steady state value.

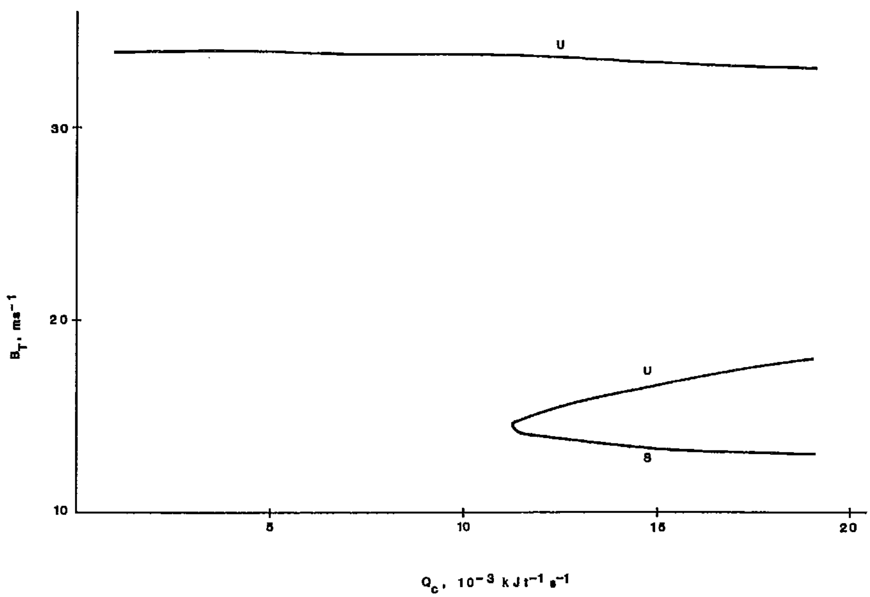

Fig. 9. Similar to Fig. 8, but with $Q_z = 8.0 \times 10^{-3}$ $kJ\,t^{-1}\,s^{-1}$.

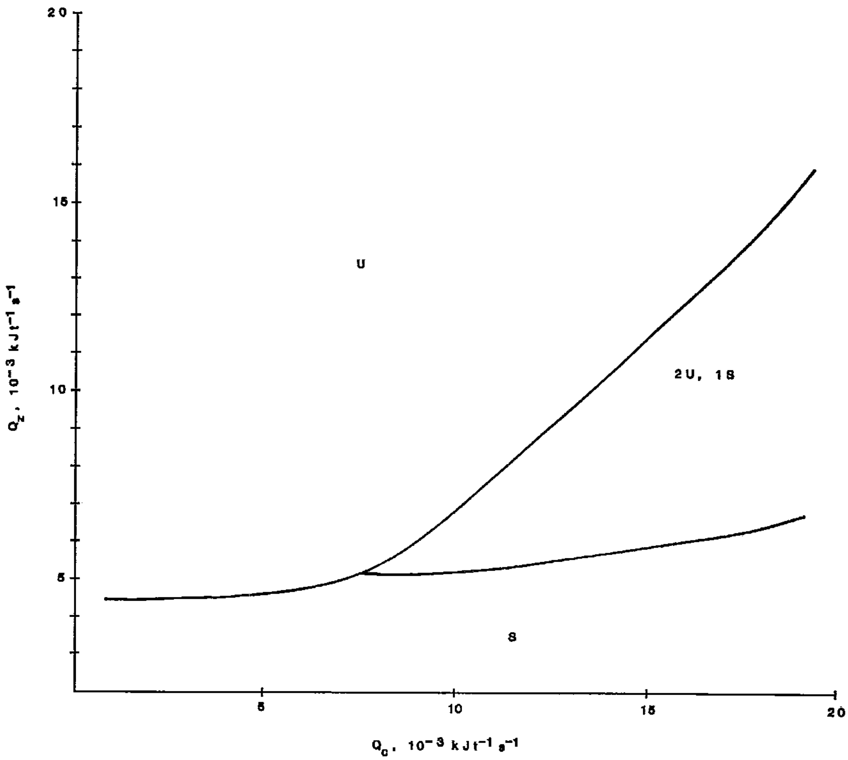

Fig. 10. An overview of the stability properties of the six component model as a function of Q_c and Q_z. For small values of the meridional heating there is stability for all values of the zonal heating. Instability exists for larger values of the meridional heating provided the zonal heating is not too large. Three solutions (one stable and two unstable) exist for large values of Q_c and moderate values of Q_z.

Although examples of the numerical integrations of the nonlinear equations have been given in an earlier paper (*Wiin-Nielsen*, 1991), we include a couple of new examples here. Fig. 11 shows a case of moderately large zonal heating, but small west-east heating. The transport of heat is shown in the upper part of the figure over one period. The middle part displays the eddy thermal kinetic energy, which is almost exactly in phase with the heat transport. The lower part of the figure indicates that the eddy kinetic energy of the vertical mean flow has its maximum about 2 days later than the maximum in the thermal kinetic energy. An extreme case is shown in Fig. 12 where the zonal heating is set at a very high value. We note the same characteristics as in the previous figure, although the time lags are somewhat larger in the latter case. However, the eddy kinetic energies have now reached a level, which are considerably larger than than those calculated from observations.

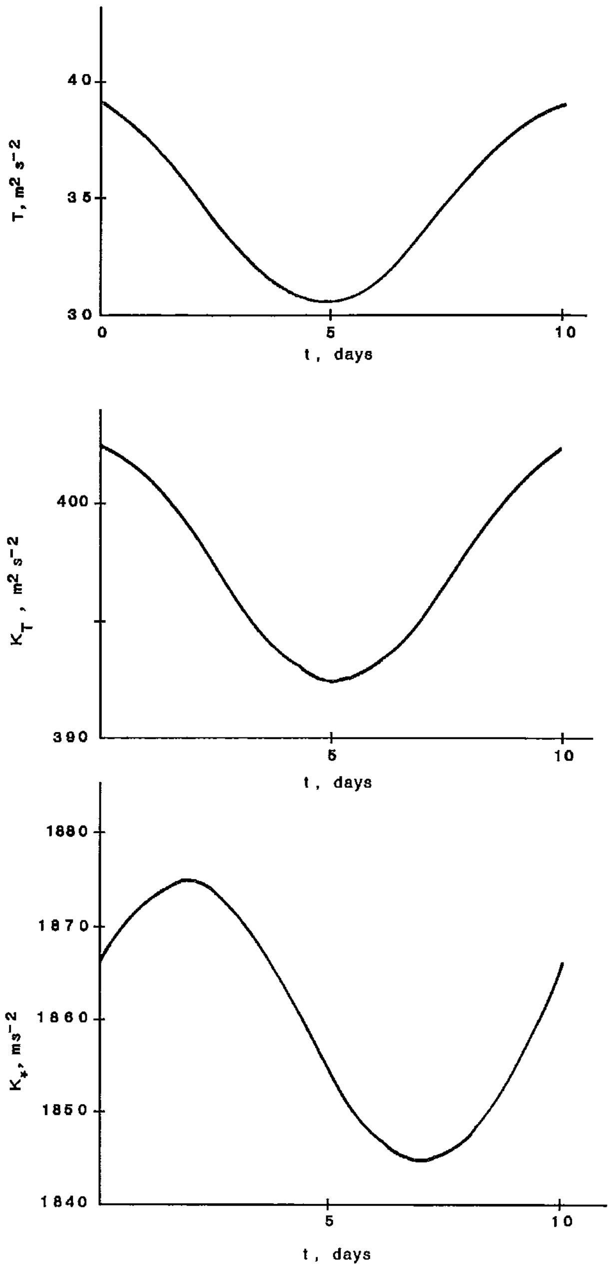

Fig. 11. The heat transport, the thermal kinetic energy and the kinetic energy of the vertical mean flow as a function of time for one period with $Q_z = 8.0 \times 10^{-3}\ kJ\,t^{-1}\,s^{-1}$ and $Q_z = 1.0 \times 10^{-3}\ kJ\,t^{-1}\,s^{-1}$. Note that the maximum of the kinetic energy of the vertical mean flow occurs about 2 days later than the maximum in the thermal kinetic energy.

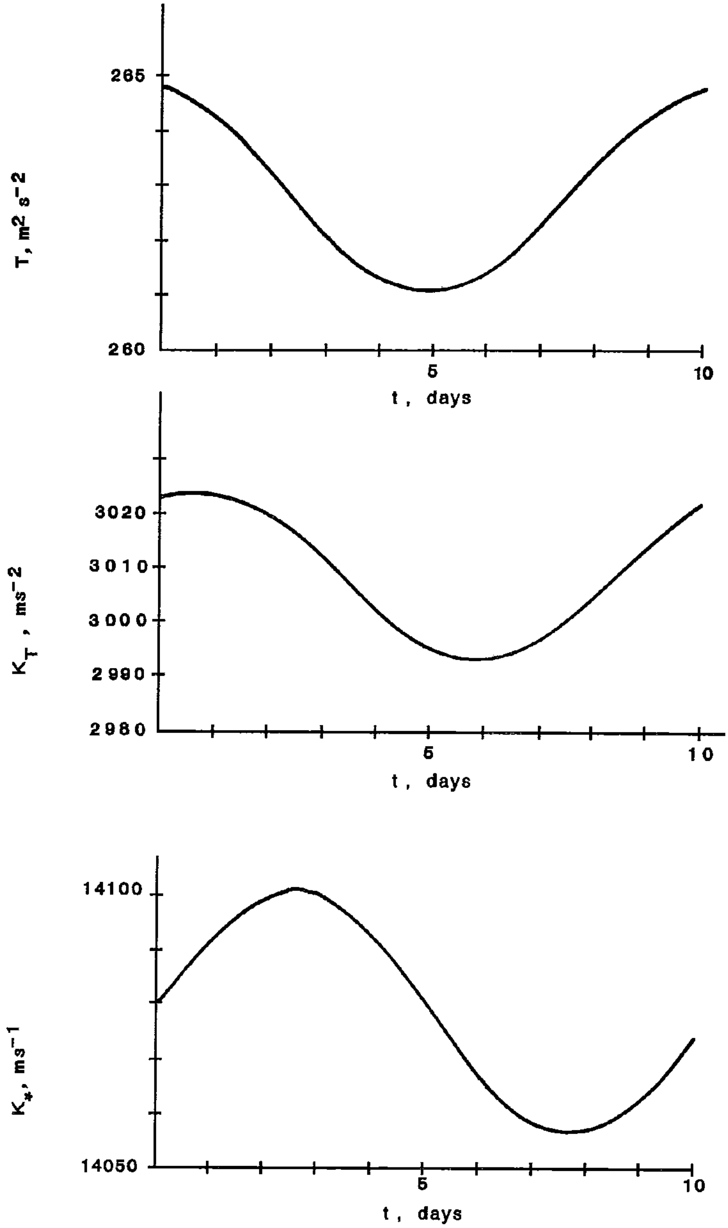

Fig. 12. The same arrangement as in Fig. 11, but with $Q_z = 3.0 \times 10^{-2}\, kJ\, t^{-1}\, s^{-1}$, while Q_c has the same value. For this large value of the meridional heating one gets a larger displacement of the maxima, but the values of the kinetic energies are unacceptably large. The periodicity is, however, maintained.

4. *The problem of long stationary waves*

The calculations in the previous sections have used a horizontal wavelength in the baroclinically unstable zone. It is of interest to explore whether or not the present nonlinear model can be used to provide a relatively simple theory of the existence of the long quasi-stationary waves in the atmosphere. The classical papers on this subject are the study of stationary waves as influenced by topography by *Charney* and *Eliassen* (1949) and *Smagorinsky*'s (1953) investigation of the long wave response to heat sources in the west-east direction. A further study of the combined effect of the two sources were carried out by *Derome* and *Wiin-Nielsen* (1971), which contains an extensive list of references to other investigations of the problem of the quasi-stationary long atmospheric waves. The theoretical studies of the problem have used a linear approach. A common assumption in all the studies is that the zonal circulation is maintained in some way by physical processes. To linearize the equations it is therefore considered permissible to adopt observed zonal winds in the advection terms in the equations. A more complete theory would specify the topography and heating fields and would provide a calculation of both the zonally averaged quantities and the structure and position of the waves. While this goal can be accomplished to a large extent by the present general circulation models of the atmosphere, we do not to the knowledge of the author possess a low order, nonlinear, analytical model which can describe both the zonal circulation and the existence, structure and position of the long stationary waves, where the orders of magnitude of the various quantities are correct, and where the structures are modeled essentially correctly however schematic they may be. While this section will not provide a solution to the problem, it will be possible to see rather easily why the present model will not suffice.

One may of course also point out that the very long waves in the atmosphere, in addition to heating and orography, are influenced by the nonlinear transfer of energy from the transient waves which therefore to a certain degree determine the intensity and position of the very long, quasi-stationary waves. In addition, the transient waves are responsible for a part of the meridional transport of sensible heat and momentum, which in turn influences the zonal current. If these processes shall be included one must require either a high order model or a good parameterization of the transport processes.

The first clue for the present model is obtained from (15). The solutions for the smaller value of the wavelength were obtained for positive values of P_2, but such solutions are out of the question, when a characteristic wavelength, say 10000 km, is used for the long waves. (16) would have to be satisfied to have $P_2 > 0$ giving a value of $B_T > 128\ m\ s^{-1}$. The only possibility is then to seek solutions, where $P_2 < 0$, but since the right hand side of (15) needs to be positive, we must also require that

$$B_T > \frac{g_z Q_z}{e_T} \tag{18}$$

which corresponds to (17) when the inequality sign is changed. The considerations can be summarized in the following inequality, where we use the values quoted above:

$$4.2673\hat{Q}_z < B_T < 128 \tag{19}$$

(19) shows that even moderately large values of $\hat{Q}_z$ will lead to values of the vertical windshear, which are large compared to the observed values. A typical value of the windshear at 30 N is $B_T = 15\ m\ s^{-1}$, which would require that $\hat{Q}_z < 3.5$. Steady state solutions satisfying (19) have been calculated, and a stability investigation of these solutions show that they are stable. However, as we can see from (14) they will have a heat transport which is negative, since $P_2 < 0$ for these solutions. The negative value of the heat transport indicates a vertical slope of the waves from west to east contrary to the observed slope. We are thus forced to conclude that the present low-order model is unsuitable for a calculation of the properties of the long quasi-stationary waves. It was of course checked that D is positive also in the case of the longer waves. We note finally that if we use values of the west-east heating which give acceptable values of the windshear, we get a much too small value of the amplitudes of the long waves.

In view of the negative results described above from a nonlinear low-order model it may be of interest to compare with the apparently more successful earlier studies. There are large differences in the formulations. We may compare with the study by *Derome* and *Wiin-Nielsen* (1971) which included forcing by topography and heating and dissipation by friction. The study is first of all linear, because the strength of the zonal circulation is prescribed. It means that there are elements of the advection of vorticity in both the vertical mean flow and in the vertical shear flow. These processes are excluded in the low-order, nonlinear model. The two models are similar in geometry, both of them employing a beta plane channel. However, it is known that the linear studies are sensitive to the width of the channel as pointed out by *Derome* (1968) who in addition to a channel width of 30 degrees latitudes used in the main part of the investigation made test calculations using wider channels. The results deteriorate noticeably when the width is increased to for example 50 degrees of latitude. It is thus possible that the apparent success is due to the choice of a narrow channel, which was selected by *Smagorinsky* (1953) as well. In the following we shall give an example which clearly indicates that the linear studies fail to give qualitatively correct results when wider channels are used.

In the example we use essentially the same two level model as the one adopted by *Derome* and *Wiin-Nielsen* (1971). We prescribe the zonal winds as constants with values taken from observations. The eddy streamfunctions are given in a form identical to the wave part of (2) and the corresponding expression for the the eddy streamfunction of the vertical mean flow. As in the remaining part of this study we shall disregard the effect of mountains and prescribe the heating in the same form as previously. The steady state equations may then be written in the form:

$$a_{11}E_* + a_{12}F_* + B_TE_T - 2a_{12}F_T = 0$$

$$-a_{12}E_* + a_{11}F_* + 2a_{12}E_T + B_TF_T = 0$$

$$a_{12}E_* + a_{32}F_* - 2a_{12}E_T + a_{34}F_T = 0$$

$$a_{32}E_* - a_{12}F_* + a_{34}E_T + 2a_{12}\, 0\, H_0 \tag{20}$$

where the coefficients have the following definitions:

$$a_{11} = B_* - \frac{\beta}{n} \qquad a_{12} = \frac{\varepsilon}{k} \qquad n = k^2 + \lambda^2$$

$$a_{32} = \left(1 - \frac{\beta}{n}\right\} \beta_T \qquad a_{34} = a_{11} + \frac{q^2}{n}\beta_* \qquad H_0 = \frac{Rq^2}{2c_pf_0}H$$

The symbols have the same meaning as in the other sections. It is seen that by adding the first and the fourth equation in the above system we get a single equation in E_* and E_T. The same procedure applied to the second and the third equation gives an equation in the remaining two variables. Solving for the variables related to the vertical mean flow we get:

$$E_* = (H_0 - (B_T + a_{34})E_T)\,/(a_{11} + a_{32})$$

$$F_* = -\,(B_T + a_{34})\,/(a_{11} + a_{32}) \tag{21}$$

From these two equations we obtain the following expression for the heat transport:

$$T = E_*F_T - E_TF_* = \frac{F_TH_0}{a_{11} + a_{32}} \tag{22}$$

When we finally solve for E_T and F_T and substitute in the expression for the heat transport we get:

$$T = \frac{H_0^2}{D_0} a_{12}(2a_{11} + B_T) \tag{23}$$

where D_0 is defined by

$$D_0 = a_{12}^2(2a_{11} + 2a_{32} + a_{34} + B_T)^2 + (a_{11}a_{34} - a_{32}B_T)^2 \tag{24}$$

From (23) we can make the observation that T is positive provided the quantity in the parenthesis is positive, since D_0 according to (24) and a_{12} according to its definition are positive quantities. T is thus positive if the inequality

$$k^2 + \lambda^2 > \frac{2\beta}{2B_* + B_T} \tag{25}$$

is satisfied. Suppose first that we consider the width to be the distance from pole to equator as we have used it in the previous section. With $B_* = 15\ m\ s^{-1}$ and $B_T = 5\ m\ s^{-1}$, which will be used also in the following examples, we find that (25) will be satisfied provided the wavelength is smaller than 7000 km, which hardly is useful in treating the long quasi-stationary waves. If we on the other hand consider the width adopted by *Derome* and *Wiin-Nielsen* (1971) (30 deg. of latitude) we find that (25) is satisfied provided the wavelength is smaller than 58000 km, which is one of the reasons that the model gives reasonable results. However, we can also see that the good behavior of the model will vanish if we select a wider channel. Suppose that we take the wavelength to be 10000 km we find from (25) that it is satisfied only if the width is less than 4360 km which corresponds to about 39 deg. of latitudes. The present example shows therefore why it has been necessary to select a channel width which is small. Otherwise the heat transport will be negative.

5. *The case of zonal forcing*

A special case of the present model was considered in a previous study (*Wiin-Nielsen*, 1990). The model was in that case restricted to zonal heating, i.e. $Q_c = 0$, which means that the four equations for the rate of change of the amplitudes (see the last two equations in (1) and (10)), respectively, become a set of homogeneous equations in the amplitudes in the case of stationary solutions. In that case one may solve the four equations for the steady state amplitudes in terms of the steady state value of the zonal windshear, compute the heat transport, and finally solve for the steady state value of B_T, noting that $B_* = 2\,B_T$. If such steady state solutions exist, they will contain both a zonal current and a wave. Another possible steady state is purely zonal, and in that case the values of the two parameters describing the zonal winds are directly related to the zonal value of the heating, i.e. to Q_z.

In deriving the formulas for the coefficients in the equations in the former investigation a mistake was made in the value for a_*. It is thus pertinent to ask if the mistake had any major influence on the results of the study. The calculations of all steady states and their stability have therefore been repeated using the procedures described above (and in more detail in *Wiin-Nielsen*, 1990). Fortunately it may be stated that only minor modifications of the numerical values were found, while the main results remain unaltered. It should be noted that the assumptions in the earlier and the present model are not quite the same. The difference is in the assumptions made in computing the surface values of wind and vorticity. In the former study it was assumed that the surface value was half of the value at level 3 (750 hPa), while the present study has assumed that the surface value may be obtained by linear extrapolation in such a way that $B_4 = B_* - 2B_T$, where B_4 is the value at 1000 hPa (close to the surface).

To illustrate the main result with the present model Fig. 13 has been prepared. Considering first the purely zonal steady states the result is that all steady states are stable except those located inside the curve on the figure. The curve is therefore as usual the curve of neutral stability. On the other hand, for the steady states containing both a zonal flow and a wave the outcome is that all steady states inside the curve on Fig. 13 are stable. The steady states outside the curve are in the latter case actually the unphysical states, which have negative kinetic energies. Such solutions appear because we have expressed the equations in terms of the transports and energies, which are second order quantities. In any case, it is found that all the unphysical states are unstable. We may thus maintain the main conclusion of the first study that a stable steady state exists for each point in the diagram. Inside the curve the steady state contain a wave, but outside the curve the steady state is zonal. Each point in the diagram is determined by the values of the wavelength and the zonal heating. It should be stressed that although the figure is similar to the standard stability diagram obtained from considering the stability of a prescribed zonal flow with an adiabatic and frictionless model, the present diagram cannot be converted to the same form, because the zonal wind speeds are determined in different ways depending on the the steady states.

Fig. 13 was prepared using the standard values of the parameters as given in the list in section 2 under (4). The numerical values leading to the curve in Fig. 13 will change if the parameter values are varied. Fig. 14 shows the result, if the numerical values of the two parameters determining the friction in the model were half as large as the standard values given in (4). We notice that while the curve occupies the same position in the wavelength coordinate, its minimum value is lower in Fig. 14 than in Fig.13. A complete parameter study has not been carried out.

The study described in this section may be considered as an extension of the classical investigations of the stability of a zonal current albeit in the case of a two level, quasi-geostrophic model. In these classical investigations one has to assume that the zonal current is maintained by the processes of heating and friction. In the present study the zonal current is created by these processes.

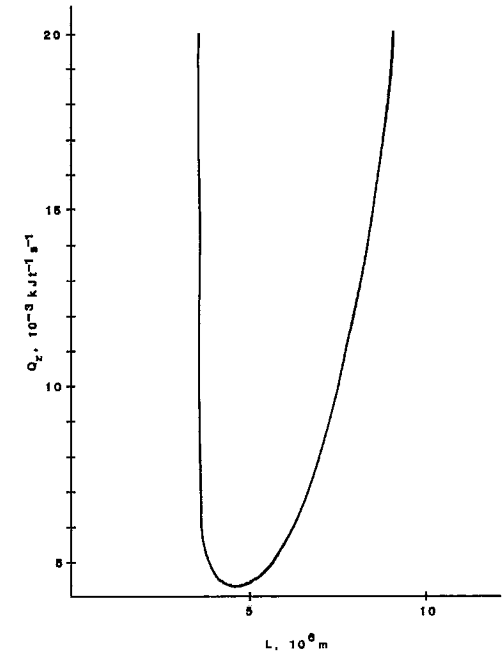

Fig. 13. The curve is the neutral curve for the case of zonal heating only in a coordinate system with the wavelength and the zonal heating along the axes. $\varepsilon = 2 \times 10^{-6}\ s^{-1}$ and $\varepsilon_T = 1.2 \times 10^{-6} s^{-1}$.

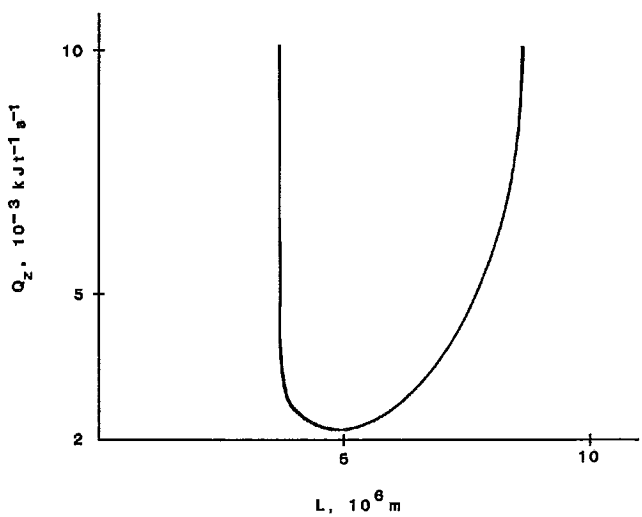

Fig. 14. Same arrangement as in Fig. 13, but with values of the two frictional values decreased to half the values.

6. *Summary and concluding remarks*

The investigation has shown that the chaotic behavior of Lorenz' low-order general circulation model disappears, when the beta-effect is added to the model. Since the effects of the Coriolis force are important in all large scale motion in the atmosphere, it is unlikely that the annual variability inherent in the Lorenz model has a counterpart in the real atmosphere.

The Lorenz model can be considered as a very special case of a low-order two level, quasi-geostrophic model. To obtain the Lorenz model from a model with six dependent variables, it is necessary to use the three equations which describe the thermal properties of the model. In addition, it is also required to specify some relation between the three variables describing the vertical mean flow and the three related to the thermal flow. It is demonstrated that the assumptions made by Lorenz regarding the waves are equivalent to assuming that the thermal wave is lagging the mean flow wave by a quarter of a wavelength. The relation between the zonal wind in the vertical mean flow and the vertical shear flow is quite open. The model used in this paper calculates the zonal wind in the vertical mean flow in such a way that the vertical mean flow is in balance from an energetical point of view, i.e. there is a balance between the energy conversion from the thermal energy to the verical mean flow and the frictional dissipation of this flow. It turns out that this ad hoc parameterization leads to a strength of the vertical mean flow of nearly twice the strength of the vertical shear flow, when the heating is rather large.

The model with six dependent variables is analysed in detail. It contains a heating in the south-north as well as the west-east directions. Internal friction as well as friction in the planetary boundary layer are included. A procedure to calculate all the stationary states of the nonlinear model is found. The stability of the stationary states is investigated. It turns out that the stationary states are stable for sufficiently small values of the meridional heating. For larger values of the meridional heating we find a single unstable stationary state as long the west-east heating is reasonably small, but for large values of the west-east heating we have three stationary states of which one is stable, namely the one with the smallest value of the zonal vertical windshear. Numerical integrations of the six equations indicate periodic solutions in the unstable cases, but an asymptotic single steady state, whenever a stable solution is present.

All the considerations summarized above deal with a single longitudinal wavelength of 5500 km. In the last section we explore the behavior of the model for a wavelength, which corresponds to the long planetary waves, say 10000 km. We find also in this case the stationary states, and they are all stable. It turns out, however, that most of them are characterized by a negative, i.e. north to south, heat transport indicating a vertical slope from west to east for increasing altitudes. Such a structure is opposite to the observed structure of the longer planetary waves. The nonlinear model is compared with the seemingly more successful linear models, in which the zonal wind fields are prescribed from observations. By selecting a relatively simple example, in which only the heating in the west-east direction, but not the topographical effects are included, it is possible to show that the success of such a model is due to a selection of an unrealistically narrow channel on the beta plane. If the width of the channel is increased, we find that also in this case we obtain a negative transport of sensible heat.

One may naturally speculate on what the requirements are, if a low order model should be able to have stationary stable solutions which are at least qualitatively correct for the long waves. The only nonlinear interaction between the zonal flow and the eddies in the six component model is the transport of sensible heat by the waves. Since the divergence of the eddy momentum transport determines the shape of the zonal currents, it is likely that the low order model should be expanded to include the momentum transport. It is known that such a model requires a minimum of 12 components. Another possibility is to parameterize both the transport of sensible heat and of momentum. This has been done by *Wiin-Nielsen* (1988). However, considering only explicit low-order models we cannot be sure that such a model is sufficient. Another possibility is that it will be required to include the interaction between the transient baroclinic waves and the long quasi-stationary waves in order to account for the position and structure of the long quasi-stationary waves. This interaction requires either the inclusion of a large number of components, in which case we would have a fully developed general circulation model, or a successful procedure for the parameterization of the effects of the transient waves in terms of the quasi-stationary long waves. However, such a procedure is unknown at present.

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